

On the Performance of Sparse Recovery via ℓ_p -minimization ($0 \leq p \leq 1$)

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Abstract

It is known that a high-dimensional sparse vector \mathbf{x}^* in \mathcal{R}^n can be recovered from low-dimensional measurements $\mathbf{y} = A\mathbf{x}^*$ where $A^{m \times n}$ ($m < n$) is the measurement matrix. In this paper, we investigate the recovering ability of ℓ_p -minimization ($0 \leq p \leq 1$) as p varies, where ℓ_p -minimization returns a vector with the least ℓ_p “norm” among all the vectors \mathbf{x} satisfying $A\mathbf{x} = \mathbf{y}$. Besides analyzing the performance of strong recovery where ℓ_p -minimization is required to recover all the sparse vectors up to certain sparsity, we also for the first time analyze the performance of “weak” recovery of ℓ_p -minimization ($0 \leq p < 1$) where the aim is to recover all the sparse vectors on one support with fixed sign pattern. When $\alpha(= \frac{m}{n}) \rightarrow 1$, we provide sharp thresholds of the sparsity ratio that differentiates the success and failure via ℓ_p -minimization. For strong recovery, the threshold strictly decreases from 0.5 to 0.239 as p increases from 0 to 1. Surprisingly, for weak recovery, the threshold is $2/3$ for all p in $[0, 1)$, while the threshold is 1 for ℓ_1 -minimization. We also explicitly demonstrate that ℓ_p -minimization ($p < 1$) can return a denser solution than ℓ_1 -minimization. For any $\alpha < 1$, we provide bounds of sparsity ratio for strong recovery and weak recovery respectively below which ℓ_p -minimization succeeds with overwhelming probability. Our bound of strong recovery improves on the existing bounds when α is large. In particular, regarding the recovery threshold, this paper argues that ℓ_p -minimization has a higher threshold with smaller p for strong recovery; the threshold is the same for all p for sectional recovery; and ℓ_1 -minimization can outperform ℓ_p -minimization for weak recovery. These are in contrast to traditional wisdom that ℓ_p -minimization, though computationally more expensive, always has better sparse recovery ability than ℓ_1 -minimization since it is closer to ℓ_0 -minimization. Finally, we provide an intuitive explanation to our findings. Numerical examples are also used to unambiguously confirm and illustrate the theoretical predictions.

I. INTRODUCTION

We consider recovering a vector \mathbf{x} in \mathcal{R}^n from an m -dimensional measurement $\mathbf{y} = A\mathbf{x}$, where $A^{m \times n}$ ($m < n$) is the measurement matrix. Obviously, given \mathbf{y} and A , $A\mathbf{x} = \mathbf{y}$ is an underdetermined linear system and admits an infinite number of solutions. However, if \mathbf{x} is sparse, i.e. it only has a small number of nonzero entries compared with its dimension, one can actually recover \mathbf{x} from \mathbf{y} . This topic is known as *compressed sensing* and draws much attention recently, for example, [7][8][16][18].

Given $\mathbf{x} \in \mathcal{R}^n$, its support T is defined as $T = \{i \in \{1, \dots, n\} : x_i \neq 0\}$. The cardinality $|T|$ of set T is the sparsity of \mathbf{x} , which also equals to the ℓ_0 norm $\|\mathbf{x}\|_0 := |\{i : x_i \neq 0\}|$. We say \mathbf{x} is ρn -sparse if $|T| = \rho n$ for some $\rho < 1$. Given the measurement \mathbf{y} and the measurement matrix A , together with the assumption that \mathbf{x} is sparse, one natural estimate of \mathbf{x} is the vector with the least ℓ_0 norm that can produce the measurement \mathbf{y} . Mathematically, to recover \mathbf{x} , we solve the following ℓ_0 -minimization problem:

$$\min_{\mathbf{x} \in \mathcal{R}^n} \|\mathbf{x}\|_0 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}. \quad (1)$$

However, (1) is combinatorial and computationally intractable, and one commonly used approach is to solve a closely related ℓ_1 -minimization problem:

$$\min_{\mathbf{x} \in \mathcal{R}^n} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}, \quad (2)$$

where $\|\mathbf{x}\|_1 := \sum_i |x_i|$. (2) is a convex problem and can be recast as a linear program, thus can be solved efficiently. Conditions under which (2) can successfully recover \mathbf{x} have been extensively studied in the literature of compressed sensing. For example, one widely known sufficient condition is the Restricted Isometry Property (RIP) [6][7][8].

Among the explosion of research on compressed sensing ([1][3][5][13][27][32][33]), recently, there has been great research interest in recovering \mathbf{x} by ℓ_p -minimization for $0 < p < 1$ ([9][10][12][14][22][29][2]) as follows,

$$\min_{\mathbf{x} \in \mathcal{R}^n} \|\mathbf{x}\|_p \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{y}. \quad (3)$$

Recall that $\|\mathbf{x}\|_p^p := (\sum_i |x_i|^p)$ for $p > 0$. Though $\|\cdot\|_p$ does not actually define a *norm* as it violates the triangular inequality, $\|\cdot\|_p^p$ follows the triangular inequality. We say \mathbf{x} can be recovered by ℓ_p -minimization if and only if it is the unique solution to (3). (3) is non-convex, and thus it is generally hard to compute the global minimum. [9][10][12] employ heuristic algorithms to compute a local minimum of (3) and show numerically that these heuristics can indeed recover sparse vectors, and the support size of these vectors can be larger than that of the vectors recoverable from ℓ_1 -minimization. Then the question is what is the relationship between the sparsity of a vector and the successful recovery with ℓ_p -minimization

($p < 1$)? How sparse should a vector be so that ℓ_p -minimization can recover it? [25] shows the sparsity up to which ℓ_p -minimization can successfully recover all the sparse vectors at least does not decrease as p decreases. [29] provides a sufficient condition for successful recovery via ℓ_p -minimization based on Restricted Isometry Constants and provides a lower bound of the support size up to which ℓ_p -minimization can recover all such sparse vectors. [22] improves this bound by considering a generalized version of RIP condition, and [4] numerically calculates this bound.

Here are the main contributions of this paper. For strong recovery where ℓ_p -minimization needs to recover all the vectors up to a certain sparsity, we provide a sharp threshold $\rho^*(p)$ of the ratio of the support size to the dimension which differentiates the success and the failure of ℓ_p -minimization when $\alpha(= \frac{m}{n}) \rightarrow 1$. This is an exact threshold compared with a lower bound of successful recovery in previous results. When ρ increases from 0 to 1, $\rho^*(p)$ decreases from 0.5 to 0.239. This coincides with the intuition that the performance of ℓ_p -minimization is improved when p decreases. When $\alpha < 1$ is fixed, we provide a positive bound $\rho^*(\alpha, p)$ for all $\alpha \in (0, 1)$ and all $p \in (0, 1]$ of strong recovery such that with a Gaussian measurement matrix $A^{m \times n}$, ℓ_p -minimization can recover all the $\rho^*(\alpha, p)n$ -sparse vectors with overwhelming probability. $\rho^*(\alpha, p)$ improves on the existing bound in large α region.

We also analyze the performance of ℓ_p -minimization for *weak* recovery where we need to recover all the sparse vectors on one support with one sign pattern. To the best of our knowledge, there is no existing result in this regard for $p < 1$. We characterize the successful weak recovery through a necessary and sufficient condition regarding the null space of the measurement matrix. When $\alpha \rightarrow 1$, we provide a sharp threshold $\rho_w^*(p)$ of the ratio of the support size to the dimension which differentiates the success and the failure of ℓ_p -minimization. The weak threshold indicates that if we would like to recover every vector over one support with size less than $\rho_w^*(p)n$ and with one sign pattern, (though the support and sign patterns are not known a priori), and we generate a random Gaussian measurement matrix independently of the vectors, then with overwhelmingly high probability, ℓ_p -minimization will recover all such vectors regardless of the amplitudes of the entries of a vector. For ℓ_1 -minimization, given a vector, if we randomly generate a Gaussian matrix and apply ℓ_1 -minimization, then its recovering ability observed in simulation exactly captures the weak recovery threshold, see [15][16]. Interestingly, we prove that the weak threshold $\rho_w^*(p)$ is $2/3$ for all $p \in [0, 1)$, and is lower than the weak threshold of ℓ_1 -minimization, which is 1. Therefore, ℓ_1 -minimization outperforms ℓ_p -minimization for all $p \in [0, 1)$ if we only need to recover sparse vectors on one support with one sign pattern. We also explicitly show that ℓ_p -minimization ($p \in (0, 1)$) can return a vector denser than the original sparse vector while ℓ_1 -minimization successfully recovers the sparse vector. Finally, for every $\alpha < 1$, we provide a positive bound $\rho_w^*(\alpha, p)$ such that

ℓ_p -minimization successfully recovers all the $\rho_w^*(\alpha, p)n$ -sparse vectors on one support with one sign pattern.

The rest of the paper is organized as follows. We introduce the null space condition of successful ℓ_p -minimization in Section II. We especially define the successful weak recovery for $p < 1$ and provide a necessary and sufficient condition. We use an example to illustrate that the solution of ℓ_1 -minimization can be sparser than that of ℓ_p -minimization ($p \in (0, 1)$). Section III provides thresholds of the sparsity ratio of the successful recovery via ℓ_p -minimization for all $p \in [0, 1]$ both in strong recovery and in weak recovery when the measurement matrix is random Gaussian matrix and $\alpha \rightarrow 1$. For $\alpha < 1$, Section IV provides bounds of sparsity ratio below which ℓ_p -minimization is successful in the strong sense and in the weak sense respectively. We compare the performance of ℓ_p -minimization ($p < 1$) and the performance of ℓ_1 -minimization in Section V and provide numerical results in Section VI. Section VII concludes the paper.

II. SUCCESSFUL RECOVERY OF ℓ_p -MINIMIZATION

We first introduce the null space characterization of the measurement matrix A to capture the successful recovery via ℓ_p -minimization ($p \in [0, 1]$). Besides the strong recovery that has been studied in [4][13][22][23][25][29][31], we especially provide a necessary and sufficient condition for the success of *weak* recovery in the sense that ℓ_p -minimization only needs to recover all the sparse vectors on one support with one sign pattern. For example, in practice, given an unknown vector to recover, we randomly generate a measurement matrix and solve the ℓ_1 -minimization problem, the simulation result of recovery performance with respect to the sparsity of the vector indeed represents the performance of weak recovery.

Given a measurement matrix $A^{m \times n}$, let $B^{n \times (n-m)}$ denote a basis of the null space of A , then we have $AB = \mathbf{0}$. Let B_i ($i \in \{1, \dots, n\}$) denote the i^{th} row of B . Let B_T denote the submatrix of B with $T \subseteq \{1, \dots, n\}$ as the set of row indices. In this paper, we will study the sparse recovery property of ℓ_p -minimization by analyzing the null space of A .

We first state the null space condition for the success of strong recovery via ℓ_p -minimization ([21][25]) in the sense that ℓ_p -minimization should recover all the sparse vectors up to a certain sparsity.

Theorem 1 ([21][25]). *\mathbf{x} is the unique solution to ℓ_p -minimization problem ($0 \leq p \leq 1$) for every vector \mathbf{x} up to ρn -sparse if and only if*

$$\|B_T \mathbf{z}\|_p^p < \|B_{T^c} \mathbf{z}\|_p^p \quad (4)$$

for every non-zero $\mathbf{z} \in \mathbb{R}^{n-m}$, and every support T with $|T| \leq \rho n$.

One important property is that if the condition (4) is satisfied for some $0 < p \leq 1$, then it is also satisfied for all $q \in [0, p]$ ([14][26]). Therefore, if ℓ_p -minimization could recover all the ρn -sparse vectors \mathbf{x} , then ℓ_q -minimization ($0 \leq q \leq p$) could also recover all the ρn -sparse vectors. Intuitively, the strong recovery performance of ℓ_q -minimization should be at least as good as that of ℓ_p -minimization when $0 \leq q < p \leq 1$.

A. Weak recovery for ℓ_p -minimization

Though ℓ_p -minimization ($p < 1$) should be at least as good as ℓ_1 -minimization for strong recovery, the argument may not be true for weak recovery.

We first state the null space condition for successful weak recovery via ℓ_1 -minimization as follows, (see [19][25][30][34][36] for this result.)

Theorem 2. *For every $\mathbf{x} \in \mathcal{R}^n$ on some support T with the same sign pattern, \mathbf{x} is always the unique solution to ℓ_1 -minimization problem (2) if and only if*

$$\|B_{T^-}\mathbf{z}\|_1 < \|B_{T^c}\mathbf{z}\|_1 + \|B_{T^+}\mathbf{z}\|_1 \quad (5)$$

holds for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ where $T^- = \{i \in T : B_i\mathbf{z}x_i < 0\}$, and $T^+ = \{i \in T : B_i\mathbf{z}x_i \geq 0\}$

Note that for every vector \mathbf{x} on a fixed support T with a fixed sign pattern, the condition to successfully recover it via ℓ_1 -minimization is the same, as stated in Theorem 2. However, the condition of successful recovery via ℓ_p -minimization ($0 \leq p < 1$) varies for different sparse vectors even if they have the same support and the same sign pattern. In other words, the recovery condition depends on the amplitudes of the entries of the vector. Here we consider the worst case scenario for weak recovery in the sense that the recovery via ℓ_p -minimization is defined to be “successful” if it can recover *all* the vectors on a fixed support with a fixed sign pattern. The null space condition for weak recovery in this definition via ℓ_1 -minimization is still the same as that in Theorem 2. We characterize the ℓ_p -minimization ($p \in (0, 1)$) case in Theorem 3 and the ℓ_0 -minimization case in Theorem 4.

Theorem 3. *Given any $p \in (0, 1)$, for all $\mathbf{x} \in \mathcal{R}^n$ on some support T with some fixed sign pattern, \mathbf{x} is always the unique solution to ℓ_p -minimization problem (3), if and only if the following condition holds:*

$$\|B_{T^-}\mathbf{z}\|_p^p \leq \|B_{T^c}\mathbf{z}\|_p^p \quad (6)$$

for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ where $T^- = \{i \in T : B_i\mathbf{z}x_i < 0\}$; moreover, if $B_{T^+}\mathbf{z} = \mathbf{0}$ where

$T^+ = \{i \in T : B_i \mathbf{z} x_i \geq 0\}$, it further holds that

$$\|B_{T^-} \mathbf{z}\|_p^p < \|B_{T^c} \mathbf{z}\|_p^p. \quad (7)$$

Proof: Necessary part. Suppose the condition fails for some \mathbf{z} , then there are two cases: either $B_{T^+} \mathbf{z} = \mathbf{0}$ or $B_{T^+} \mathbf{z} \neq \mathbf{0}$.

First consider the case $B_{T^+} \mathbf{z} = \mathbf{0}$, then we have $\|B_{T^-} \mathbf{z}\|_p^p \geq \|B_{T^c} \mathbf{z}\|_p^p$. Define a vector \mathbf{x} as follows. Let $x_i = 0$ for every i in T^c , let $x_i = -B_i \mathbf{z}$ for every i in T^- . Let x_i be any value with the fixed sign for every i in T^+ . Then according to the definition of \mathbf{x} , we have

$$\begin{aligned} & \|\mathbf{x} + B\mathbf{z}\|_p^p \\ &= \|\mathbf{x}_{T^-} + B_{T^-} \mathbf{z}\|_p^p + \|\mathbf{x}_{T^+} + B_{T^+} \mathbf{z}\|_p^p + \|B_{T^c} \mathbf{z}\|_p^p \\ &= 0 + \|\mathbf{x}_{T^+}\|_p^p + \|B_{T^c} \mathbf{z}\|_p^p \\ &= \|\mathbf{x}\|_p^p - \|\mathbf{x}_{T^-}\|_p^p + \|B_{T^c} \mathbf{z}\|_p^p \\ &= \|\mathbf{x}\|_p^p - \|B_{T^-} \mathbf{z}\|_p^p + \|B_{T^c} \mathbf{z}\|_p^p \\ &\leq \|\mathbf{x}\|_p^p. \end{aligned}$$

Since $\|\mathbf{x} + B\mathbf{z}\|_p^p \leq \|\mathbf{x}\|_p^p$, (3) cannot successfully recover \mathbf{x} , which is a contradiction.

Secondly, consider the case $B_{T^+} \mathbf{z} \neq \mathbf{0}$. Then $\|B_{T^-} \mathbf{z}\|_p^p > \|B_{T^c} \mathbf{z}\|_p^p$. Let $\delta = \|B_{T^-} \mathbf{z}\|_p^p - \|B_{T^c} \mathbf{z}\|_p^p > 0$. Define a vector \mathbf{x} as follows. Let $x_i = 0$ for every i in T^c , let $x_i = -B_i \mathbf{z}$ for every i in T^- . For every i in T^+ , since $p \in (0, 1)$, we can pick x_i with $|x_i|$ large enough such that $\|\mathbf{x}_{T^+} + B_{T^+} \mathbf{z}\|_p^p - \|\mathbf{x}_{T^+}\|_p^p < \frac{\delta}{2}$. Then

$$\begin{aligned} \|\mathbf{x} + B\mathbf{z}\|_p^p &= 0 + \|\mathbf{x}_{T^+} + B_{T^+} \mathbf{z}\|_p^p + \|B_{T^c} \mathbf{z}\|_p^p \\ &< \|\mathbf{x}_{T^+}\|_p^p + \frac{\delta}{2} + \|B_{T^c} \mathbf{z}\|_p^p \\ &= \|\mathbf{x}_{T^+}\|_p^p + \frac{\delta}{2} + \|B_{T^-} \mathbf{z}\|_p^p - \delta \\ &= \|\mathbf{x}\|_p^p - \frac{\delta}{2}. \end{aligned}$$

Thus $\|\mathbf{x} + B\mathbf{z}\|_p^p < \|\mathbf{x}\|_p^p$, \mathbf{x} is not a solution to (3), which is also a contradiction.

Sufficient part. Assume the null space condition holds, then for any \mathbf{x} on support T with fixed signs,

and any non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, we have

$$\begin{aligned}
& \|\mathbf{x} + B\mathbf{z}\|_p^p \\
&= \|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p + \|\mathbf{x}_{T^-} + B_{T^-}\mathbf{z}\|_p^p + \|B_{T^c}\mathbf{z}\|_p^p \\
&\geq \|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p + \|\mathbf{x}_{T^-}\|_p^p - \|B_{T^-}\mathbf{z}\|_p^p + \|B_{T^c}\mathbf{z}\|_p^p,
\end{aligned} \tag{8}$$

where the inequality follows from the triangular property that $|\mathbf{x}_i + B_i\mathbf{z}|^p \geq |\mathbf{x}_i|^p - |B_i\mathbf{z}|^p$ holds for all i and all $p \in (0, 1)$.

If $B_{T^+}\mathbf{z} \neq \mathbf{0}$, then $\|\mathbf{x}_{T^+} + B_{T^+}\mathbf{z}\|_p^p > \|\mathbf{x}_{T^+}\|_p^p$ since $B_i\mathbf{z} \neq 0$ for some i , and $B_i\mathbf{z}$ and x_i have the same sign. Since we also have $\|B_{T^-}\mathbf{z}\|_p^p \leq \|B_{T^c}\mathbf{z}\|_p^p$, therefore (8) $> \|\mathbf{x}\|_p^p$. If $B_{T^+}\mathbf{z} = \mathbf{0}$, then $\|B_{T^-}\mathbf{z}\|_p^p < \|B_{T^c}\mathbf{z}\|_p^p$ from assumption, therefore we also have (8) $> \|\mathbf{x}\|_p^p$. Thus, $\|\mathbf{x} + B\mathbf{z}\|_p^p > \|\mathbf{x}\|_p^p$ for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, then \mathbf{x} is the solution to (3). \blacksquare

Similarly, the null space condition for the weak recovery of ℓ_0 -minimization is as follows, we skip its proof as it is similar to that of Theorem 3.

Theorem 4. *For all $\mathbf{x} \in \mathcal{R}^n$ on one support T with the same sign pattern, \mathbf{x} is always the unique solution to ℓ_0 -minimization problem (1), if and only if*

$$\|B_{T^-}\mathbf{z}\|_0 < \|B_{T^c}\mathbf{z}\|_0 \tag{9}$$

for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ where $T^- = \{i \in T : B_i\mathbf{z}x_i < 0\}$.

For the strong recovery, the null space conditions of ℓ_1 -minimization and ℓ_p -minimization ($0 \leq p < 1$) share the same form (4), and if (4) holds for some $p \leq 1$, it also holds for all $q \in [0, p]$. However, for recovery of sparse vectors on one support with one sign pattern, from Theorem 2, 3 and 4, we know that although the conditions of ℓ_p -minimization ($0 < p < 1$) and ℓ_0 -minimization share a similar form in (6), (7) and (9), the condition of ℓ_1 -minimization has a very different form in (5). Moreover, if (6) holds for some $p \in (0, 1)$, it does not necessarily hold for some $q \in (0, p)$. Therefore the way that the performance of weak recovery changes over p may be quite different from the way that the performance of strong recovery changes over p . Moreover, the performance of weak recovery of ℓ_1 may be significantly different from that of ℓ_p -minimization for $p \in (0, 1)$. We will further discuss this issue.

B. The solution of ℓ_1 -minimization can be sparser than that of ℓ_p -minimization ($p \in (0, 1)$)

ℓ_p -minimization ($p \in (0, 1)$) may not perform as well as ℓ_1 -minimization in some cases, for example in the weak recovery which we will discuss in Section III and Section IV. Here we employ a numerical

example to illustrate that in certain cases ℓ_1 -minimization can recover the sparse vector while ℓ_p -minimization ($p \in (0, 1)$) cannot, and the solution of ℓ_p -minimization is denser than the original sparse vector.

Example 1. ℓ_p -minimization returns a denser solution than ℓ_1 -minimization.

Let the measurement matrix A be a $(6k-1) \times 6k$ matrix with $\beta \in \mathcal{R}^{6k}$ as a basis of its null space, and $\beta_i = 1$ for all $i \in \{1, \dots, k\}$, $\beta_i = -1$ for all $i \in \{k+1, \dots, 2k\}$, and $\beta_i = 1/64$ for all $i \in \{2k+1, \dots, 6k\}$. According to Theorem 1, one can calculate that ℓ_1 -minimization can recover all the $(\lceil \frac{33}{32}k \rceil - 1)$ -sparse vectors in \mathcal{R}^{6k} , and $\ell_{0.5}$ -minimization can recover all the $(\lceil \frac{5}{4}k \rceil - 1)$ -sparse vectors in \mathcal{R}^{6k} . Therefore, in terms of strong recovery, $\ell_{0.5}$ -minimization has a better performance than ℓ_1 -minimization as it can recover all the vectors up to a higher sparsity.

Now consider the “weak” recovery as to recover all the nonnegative vectors on support $T = \{1, \dots, 2k\}$. According to Theorem 2 and Theorem 3, one can check that ℓ_1 -minimization can indeed recover all the nonnegative vectors on support T , however, $\ell_{0.5}$ -minimization fails to recover some vectors in this case. For example, consider a $2k$ -sparse vector \mathbf{x}^* with $x_i^* = 9$ for all $i \in \{1, \dots, k\}$, $x_i^* = 1$ for all $i \in \{k+1, \dots, 2k\}$, and $x_i^* = 0$ for all $i \in \{2k+1, \dots, 6k\}$. One can check that among all the vectors $\mathbf{x} = \mathbf{x}^* + h\beta$, $\forall h \in \mathcal{R}$, which are the solutions to $A\mathbf{x} = A\mathbf{x}^*$, \mathbf{x}^* has the least ℓ_1 norm, therefore \mathbf{x}^* is the solution to (2) and can be successfully recovered via ℓ_1 -minimization. Now consider $\ell_{0.5}$ -minimization, we have $\|\mathbf{x}^*\|_{0.5}^{0.5} = 4k$. Consider the nonnegative $5k$ -sparse vector $\mathbf{x}' = \mathbf{x}^* + \beta$ with $x_i' = 10$ for all $i \in \{1, \dots, k\}$, $x_i' = 0$ for all $i \in \{k+1, \dots, 2k\}$, and $x_i' = 1/64$ for all $i \in \{2k+1, \dots, 6k\}$. We have $A\mathbf{x}' = A\mathbf{x}^*$, and one can check that $\|\mathbf{x}'\|_{0.5}^{0.5} = (\sqrt{10} + 0.5)k < \|\mathbf{x}^*\|_{0.5}^{0.5}$ for all $k \geq 2$. Moreover, with a little calculation one can prove that \mathbf{x}' is indeed the solution to (3). Thus, the solution of $\ell_{0.5}$ -minimization is a $5k$ -sparse vector although the original vector \mathbf{x}^* is only $2k$ -sparse. Therefore $\ell_{0.5}$ -minimization fails to recover some nonnegative $2k$ -sparse vector \mathbf{x}^* while \mathbf{x}^* is the solution to ℓ_1 -minimization, and the solution of $\ell_{0.5}$ -minimization is denser than the original vector \mathbf{x}^* .

III. RECOVERY THRESHOLDS WHEN $\lim_{n \rightarrow \infty} \frac{m}{n} \rightarrow 1$

In this paper we focus on the case that each entry of the measurement matrix A is drawn from standard Gaussian distribution. Since A has i.i.d. $\mathcal{N}(0, 1)$ entries, the null space of A is rotationally invariant, thus there exists a basis $B^{n \times (n-m)}$ of the null space of A such that $AB = \mathbf{0}$ and B has i.i.d. $\mathcal{N}(0, 1)$ entries, please refer to [8][35] for details.

We first focus on the case that $\alpha = \frac{m}{n} \rightarrow 1$ and provide recovery thresholds of ℓ_p -minimization for every $p \in [0, 1]$. we consider two types of thresholds: one in the *strong* sense as we require ℓ_p -minimization to

recover *all* ρn -sparse vectors (Section III-A), one in the *weak* sense as we only require ℓ_p -minimization to recover *all the vectors on a certain support with a certain sign pattern* (Section III-B). We call it a threshold as for any sparsity below that threshold, ℓ_p -minimization can recover all the sparse vectors either in the strong sense or the weak sense, and for any sparsity above that threshold, ℓ_p -minimization fails to recover some sparse vector. These thresholds can be viewed as the limiting behavior of ℓ_p -minimization, since for any constant $\alpha < 1$, the recovery thresholds of ℓ_p -minimization would be no greater than the ones provided here.

A. Strong Recovery

In this section, for given p , when $\alpha \rightarrow 1$, we shall provide a threshold $\rho^*(p)$ for *strong recovery* such that for any $\rho < \rho^*(p)$, ℓ_p -minimization (3) can recover *all* ρn -sparse vectors \mathbf{x} with overwhelming probability. Our technique here stems from [20], which only focuses on the strong recovery of ℓ_1 -minimization.

We have already discussed in Section II that the performance of ℓ_q -minimization should be no worse than ℓ_p -minimization for strong recovery when $0 \leq q < p \leq 1$. Although there are results about bound of the sparsity below which ℓ_p -minimization can recover all the sparse vectors, no existing result has explicitly calculated the recovery threshold of ℓ_p -minimization for $p < 1$ which differentiates the success and failure of ℓ_p -minimization. To this end, we will first define $\rho^*(p)$ in the following lemma, and then prove that $\rho^*(p)$ is indeed the threshold of strong recovery in later part.

Lemma 1. *Let X_1, X_2, \dots, X_n be i.i.d $\mathcal{N}(0, 1)$ random variables and let Y_1, Y_2, \dots, Y_n be the sorted ordering (in non-increasing order) of $|X_1|^p, |X_2|^p, \dots, |X_n|^p$ for some $p \in (0, 1]$. For a $\rho > 0$, define S_ρ as $\sum_{i=1}^{\lceil \rho n \rceil} Y_i$. Let S denote $E[S_1]$, the expected value of S_1 . Then there exists a constant $\rho^*(p)$ such that $\lim_{n \rightarrow \infty} \frac{E[S_{\rho^*}]}{S} = \frac{1}{2}$.*

Proof: Let $X \sim \mathcal{N}(0, 1)$ and let $Z = |X|$. Let $f(z)$ and $F(z)$ denote the p.d.f. and c.d.f. of Z respectively. Then

$$\begin{aligned} f(z) &= \sqrt{2/\pi} e^{-\frac{1}{2}z^2}, \quad \text{if } z \geq 0, \\ &= 0, \quad \text{if } z < 0. \end{aligned} \tag{10}$$

$$\begin{aligned} F(z) &= \text{erf}(z/\sqrt{2}) = \int_0^z \sqrt{2/\pi} e^{-\frac{1}{2}x^2} dx, \quad \text{if } z \geq 0, \\ &= 0, \quad \text{if } z < 0. \end{aligned} \tag{11}$$

Define $g(t) = \int_t^\infty z^p f(z) dz$. g is continuous and decreasing in $[0, \infty]$, and $g(0) = E[Z^p] = \frac{S}{n}$, $\lim_{t \rightarrow \infty} g(t) = 0$. Then there exists z^* such that $g(z^*) = \frac{g(0)}{2}$, i.e.

$$\int_0^{z^*} x^p f(x) dx - \int_{z^*}^\infty x^p f(x) dx = 0. \quad (12)$$

Define

$$\rho^* = 1 - F(z^*). \quad (13)$$

We claim ρ^* has the desired property.

Let $T_t = \sum_{i: Y_i \geq t^p} Y_i$. Then $E[T_{z^*}] = ng(z^*)$. Since $E[|T_{z^*} - S_{\rho^*}|]$ is bounded by $O(\sqrt{n})$, and $S = ng(0)$, thus $\lim_{n \rightarrow \infty} \frac{E[S_{\rho^*}]}{S} = \frac{1}{2}$. ■

Proposition 1. *The function $\rho^*(p)$ is strictly decreasing in p on $(0, 1]$.*

Proof: From the definition of z^* in (12), we have

$$H(z^*, p) := \int_0^{z^*} x^p f(x) dx - \int_{z^*}^\infty x^p f(x) dx = 0, \quad (14)$$

where $f(\cdot)$ and $F(\cdot)$ are defined in (10) and (11). From the Implicit Function Theorem,

$$\frac{dz^*}{dp} = -\frac{\frac{\partial H}{\partial p}}{\frac{\partial H}{\partial z^*}} = -\frac{\int_0^{z^*} x^p (\ln x) f(x) dx - \int_{z^*}^\infty x^p (\ln x) f(x) dx}{2z^{*p} f(z^*)}.$$

From (13), we have $\frac{d\rho^*}{dz^*} = -f(z^*)$. From the chain rule, we know $\frac{d\rho^*}{dp} = \frac{d\rho^*}{dz^*} \frac{dz^*}{dp}$, thus

$$\frac{d\rho^*}{dp} = \frac{\int_0^{z^*} x^p (\ln x) f(x) dx - \int_{z^*}^\infty x^p (\ln x) f(x) dx}{2z^{*p}} \quad (15)$$

Note that

$$\begin{aligned} \int_0^{z^*} x^p (\ln x) f(x) dx &< \int_0^{z^*} x^p (\ln z^*) f(x) dx \\ &= \int_{z^*}^\infty x^p (\ln z^*) f(x) dx \\ &< \int_{z^*}^\infty x^p (\ln x) f(x) dx, \end{aligned} \quad (16)$$

where the equality follows from (14). Then the numerator of (15) is less than 0 from (16), thus $\frac{d\rho^*}{dp} < 0$. ■

We plot ρ^* against p numerically in Fig. 1. $\rho^*(p)$ goes to $\frac{1}{2}$ as p tends to zero. Note that $\rho^*(1) = 0.239\dots$, which coincides with the result in [20].

Now we proceed to prove that ρ^* is the threshold of successful recovery with ℓ_p minimization for p in $(0, 1]$. First we state the concentration property of S_ρ in the following lemma.

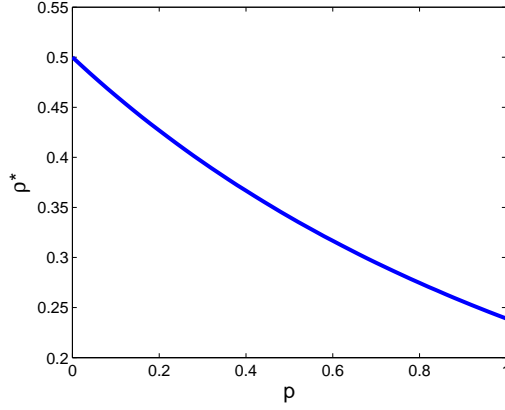


Fig. 1. Threshold ρ^* of successful recovery with ℓ_p -minimization

Lemma 2. For any $p \in (0, 1]$, let $X_1, \dots, X_n, Y_1, \dots, Y_n, S_\rho$ and S be as above. For any $\rho > 0$ and any $\delta > 0$, there exists a constant $c_1 > 0$ such that when n is large enough, with probability at least $1 - 2e^{-c_1 n}$, $|S_\rho - E[S_\rho]| \leq \delta S$.

Proof: Let $\mathbf{X} = [X_1, \dots, X_n]^T$. If two vectors \mathbf{X} and \mathbf{X}' only differ in co-ordinate i , then for any p , $|S_\rho(\mathbf{X}) - S_\rho(\mathbf{X}')| \leq ||X_i|^p - |X'_i|^p|$. Thus for any \mathbf{X} and \mathbf{X}' ,

$$|S_\rho(\mathbf{X}) - S_\rho(\mathbf{X}')| \leq \sum_{i: X_i \neq X'_i} ||X_i|^p - |X'_i|^p|.$$

Since $||X_i|^p - |X'_i|^p| \leq |X_i - X'_i|^p$ for all $p \in (0, 1]$,

$$|S_\rho(\mathbf{X}) - S_\rho(\mathbf{X}')| \leq \sum_i |X_i - X'_i|^p. \quad (17)$$

From the isoperimetric inequality for the Gaussian measure [28], for any set A with measure at least a half, the set $A_t = \{\mathbf{x} \in \mathcal{R}^n : d(\mathbf{x}, A) \leq t\}$ has measure at least $1 - e^{-t^2/2}$, where $d(\mathbf{x}, A) = \inf_{\mathbf{y} \in A} \|\mathbf{x} - \mathbf{y}\|_2$. Let M_ρ be the median value of $S_\rho = S_\rho(\mathbf{X})$. Define set $A = \{\mathbf{x} \in \mathcal{R}^n : S_\rho(\mathbf{x}) \leq M_\rho\}$, then

$$P(d(\mathbf{x}, A) \leq t) \geq 1 - e^{-t^2/2}.$$

We claim that $d(\mathbf{x}, A) \leq t$ implies that $S_\rho(\mathbf{x}) \leq M_\rho + n^{(1-p/2)}t^p$. If $\mathbf{x} \in A$, then $S_\rho(\mathbf{x}) \leq M_\rho$, thus the claim holds as $n^{1-p/2}t^p$ is nonnegative. If $\mathbf{x} \notin A$, then there exists $\mathbf{x}' \in A$ such that $\|\mathbf{x} - \mathbf{x}'\|_2 \leq t$. Let

$u_i = 1$ for all i and let $v_i = |x_i - x'_i|^p$. From Hölder's inequality,

$$\begin{aligned} \sum_i |x_i - x'_i|^p &\leq \left(\sum_i |u_i|^{2/(2-p)} \right)^{1-p/2} \left(\sum_i |v_i|^{2/p} \right)^{p/2} \\ &\leq n^{(1-p/2)} (t^2)^{p/2} = n^{(1-p/2)} t^p \end{aligned} \quad (18)$$

From (17) and (18), $|S_\rho(\mathbf{x}) - S_\rho(\mathbf{x}')| \leq n^{(1-p/2)} t^p$. Since $\mathbf{x} \notin A$ and $\mathbf{x}' \in A$, then $S_\rho(\mathbf{x}) > M_\rho \geq S_\rho(\mathbf{x}')$. Thus $S_\rho(\mathbf{x}) \leq M_\rho + n^{(1-p/2)} t^p$, which verifies our claim. Then

$$P(S_\rho(\mathbf{x}) \leq M_\rho + n^{(1-p/2)} t^p) \geq P(d(\mathbf{x}, A) \leq t) \geq 1 - e^{-t^2/2}. \quad (19)$$

Similarly,

$$P(S_\rho(\mathbf{x}) \geq M_\rho - n^{(1-p/2)} t^p) \geq 1 - e^{-t^2/2}. \quad (20)$$

Combining (19) and (20),

$$P(|S_\rho(x) - M_\rho| \geq n^{(1-p/2)} t^p) \leq 2e^{-t^2/2}. \quad (21)$$

The difference of $E[S_\rho]$ and M_ρ can be bounded as follows,

$$\begin{aligned} |E[S_\rho] - M_\rho| &\leq E[|S_\rho - M_\rho|] \\ &= \int_0^\infty P(|S_\rho(x) - M_\rho| \geq y) dy \\ &\leq \int_0^\infty 2e^{-\frac{1}{2}y^{\frac{2}{p}} n^{(1-\frac{2}{p})}} dy \\ &= n^{(1-\frac{2}{p})} \int_0^\infty 2e^{-\frac{1}{2}s^{\frac{2}{p}}} ds \end{aligned}$$

Note that $c := \int_0^\infty 2e^{-\frac{1}{2}s^{(2/p)}} ds$ is a finite constant for all $p \in (0, 1]$. As $p > 0$ and $S = nE[|x_i|^p]$, thus for any $\delta > 0$, $cn^{(1-\frac{2}{p})} < \frac{\delta}{2}S$ when n is large enough.

Let $t = \left(\frac{1}{2}\delta S n^{(\frac{p}{2}-1)}\right)^{\frac{1}{p}} = \left(\frac{1}{2}\delta E[|x_i|^p]\right)^{\frac{1}{p}} \sqrt{n}$, from (21) with probability at least $1 - 2e^{-\frac{1}{2}(\frac{1}{2}\delta E[|x_i|^p])^{\frac{2}{p}} n}$, $|S_\rho - M_\rho| < \frac{1}{2}\delta S$. Thus $|S_\rho - E[S_\rho]| \leq |S_\rho - M_\rho| + |M_\rho - E[S_\rho]| < \delta S$ with probability at least $1 - 2e^{-c_1 n}$ for some constant c_1 . ■

Corollary 1. *For any $\rho < \rho^*$, there exists a $\delta > 0$ and a constant $c_2 > 0$ such that when n is large enough, with probability at least $1 - 2e^{-c_2 n}$, $S_\rho \leq (\frac{1}{2} - \delta)S$.*

Proof: When $\rho < \rho^*$,

$$\begin{aligned} E[S_\rho] &= E[S_{\rho^*}] - \sum_{i=\lceil \rho n \rceil + 1}^{\lceil \rho^* n \rceil} E[|X_i|^p] \\ &\leq E[S_{\rho^*}] - (\lceil \rho^* n \rceil - \lceil \rho n \rceil) E[|X_i|^p] \end{aligned}$$

Then $E[S_\rho]/S \leq \frac{1}{2} - 2\delta$ for a suitable δ as $S = nE[|X_i|^p]$. The result follows by combining the above with Lemma 2. \blacksquare

Corollary 2. *For any $\epsilon > 0$, there exists a constant $c_3 > 0$ such that when n is large enough, with probability at least $1 - 2e^{-c_3 n}$, it holds that $(1 - \epsilon)S \leq S_1 \leq (1 + \epsilon)S$.*

The above two corollaries indicate that with overwhelming probability the sum of the largest $\lceil \rho n \rceil$ terms of Y_i 's is less than half of the total sum S_1 if $\rho < \rho^*$. The following lemma extends the result to every vector $B\mathbf{z}$ where matrix $B^{n \times (n-m)}$ has i.i.d. Gaussian entries and \mathbf{z} is any non-zero vector in \mathcal{R}^{n-m} .

Lemma 3. *For any $0 < p \leq 1$, given any $\rho < \rho^*(p)$, there exist constants $0 < c_4 < 1$, $c_5 > 0$, $\delta > 0$ such that when $\alpha = \frac{m}{n} > c_4$ and n is large enough, with probability at least $1 - e^{-c_5 n}$, an $n \times (n-m)$ matrix B with i.i.d. $\mathcal{N}(0, 1)$ entries has the following property: for every non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ and every subset $T \subseteq \{1, \dots, n\}$ with $|T| \leq \rho n$, $\|B_{T^c}\mathbf{z}\|_p^p - \|B_T\mathbf{z}\|_p^p \geq \delta S \|\mathbf{z}\|_2^p$.*

Proof: For any given $\gamma > 0$, there exists a γ -net Σ in \mathcal{R}^{n-m} of cardinality less than $(1 + \frac{2}{\gamma})^{n-m}$ ([28]). A γ -net Σ is a set of points in \mathcal{R}^{n-m} such that $\|\mathbf{v}^k\|_2 = 1$ for all \mathbf{v}^k in Σ and for any $\mathbf{z} \in \mathcal{R}^{n-m}$ with $\|\mathbf{z}\|_2 = 1$, there exists some \mathbf{v}^k such that $\|\mathbf{z} - \mathbf{v}^k\|_2 \leq \gamma$.

Since B has i.i.d. $\mathcal{N}(0, 1)$ entries, then $B\mathbf{v}^k$ has n i.i.d. $\mathcal{N}(0, 1)$ entries for every \mathbf{v}^k . From Corollary 1 and 2, we know that given any $\rho < \rho^*$, for some $\delta > 0$ and for every $\epsilon > 0$, there exists $c_2 > 0$ and c_3 such that with probability at least $1 - 2e^{-c_2 n} - 2e^{-c_3 n}$, we have

$$S_\rho(A\mathbf{v}^k) \leq \left(\frac{1}{2} - \delta\right)S \quad (22)$$

and

$$(1 - \epsilon)S \leq S_1(A\mathbf{v}^k) \leq (1 + \epsilon)S \quad (23)$$

both hold for a vector \mathbf{v}^k in Σ . Then applying union bound, we know that (22) and (23) hold for all vectors in Σ with probability at least

$$1 - (1 + 2/\gamma)^{n-m}(2e^{-c_2 n} + 2e^{-c_3 n}). \quad (24)$$

Let $\alpha = m/n$, then as long as $\alpha > c_4 := 1 - \frac{\min(c_2, c_3)}{\ln(1+2/\gamma)}$, then (24) $\geq 1 - e^{-c_5 n}$ for some constant $c_5 > 0$.

For any \mathbf{z} such that $\|\mathbf{z}\|_2 = 1$, there exists \mathbf{v}_0 in Σ such that $\|\mathbf{z} - \mathbf{v}_0\|_2 \triangleq \gamma_1 \leq \gamma$. Let \mathbf{z}_1 denote $\mathbf{z} - \mathbf{v}_0$, then $\|\mathbf{z}_1 - \gamma_1 \mathbf{v}_1\|_2 \triangleq \gamma_2 \leq \gamma_1 \gamma \leq \gamma^2$ for some \mathbf{v}_1 in Σ . Repeating this process, we have

$$\mathbf{z} = \sum_{j \geq 0} \gamma_j \mathbf{v}_j \quad (25)$$

where $\gamma_0 = 1$, $\gamma_j \leq \gamma^j$ and $\mathbf{v}_j \in \Sigma$. Thus for any $\mathbf{z} \in \mathcal{R}^{n-m}$, we have $\mathbf{z} = \|\mathbf{z}\|_2 \sum_{j \geq 0} \gamma_j \mathbf{v}_j$.

For any index set T with $|T| \leq \rho n$,

$$\begin{aligned} \|B_T \mathbf{z}\|_p^p &= \|\mathbf{z}\|_2^p \sum_{j \geq 0} \gamma_j \|B_T \mathbf{v}_j\|_p^p \\ &\leq \|\mathbf{z}\|_2^p \sum_{j \geq 0} \gamma^{jp} \|B_T \mathbf{v}_j\|_p^p \\ &\leq S \|\mathbf{z}\|_2^p \frac{1 - 2\delta}{2(1 - \gamma^p)}, \end{aligned}$$

$$\begin{aligned} \|B\mathbf{z}\|_p^p &= \|\mathbf{z}\|_2^p \sum_{j \geq 0} \gamma_j \|B\mathbf{v}_j\|_p^p \\ &\geq \|\mathbf{z}\|_2^p (\|B\mathbf{v}_0\|_p^p - \sum_{j \geq 1} \gamma_j^p \|B\mathbf{v}_j\|_p^p) \\ &\geq \|\mathbf{z}\|_2^p (\|B\mathbf{v}_0\|_p^p - \sum_{j \geq 1} \gamma^{jp} \|B\mathbf{v}_j\|_p^p) \\ &\geq \|\mathbf{z}\|_2^p ((1 - \epsilon)S - \sum_{j \geq 1} \gamma^{jp} (1 + \epsilon)S) \\ &\geq S \|\mathbf{z}\|_2^p \frac{1 - 2\gamma^p - \epsilon}{1 - \gamma^p} \end{aligned}$$

Thus $\|B_{T^c} \mathbf{z}\|_p^p - \|B_T \mathbf{z}\|_p^p \geq S \|\mathbf{z}\|_2^p \frac{2\delta - 2\gamma^p - \epsilon}{1 - \gamma^p}$. For a given δ , we can pick γ and ϵ small enough such that $\|B_{T^c} \mathbf{z}\|_p^p - \|B_T \mathbf{z}\|_p^p \geq \delta S \|\mathbf{z}\|_2^p$. ■

We can now establish one main result regarding the threshold of successful recovery via ℓ_p -minimization.

Theorem 5. *For any $0 < p \leq 1$, given any $\rho < \rho^*(p)$, there exist constants $0 < c_4 < 1$, $c_5 > 0$ such that when $\alpha > c_4$ and n is large enough, with probability at least $1 - e^{-c_5 n}$, an $m \times n$ matrix A with i.i.d. $\mathcal{N}(0, 1)$ entries has the following property: for every $\mathbf{x} \in \mathcal{R}^n$ with its support T satisfying $|T| \leq \rho n$, \mathbf{x} is the unique solution to the ℓ_p -minimization problem (3).*

Proof: Lemma 3 indicates that $\sum_{i \in T^c} |(B\mathbf{z})_i|^p - \sum_{i \in T} |(B\mathbf{z})_i|^p \geq \delta S \|\mathbf{z}\|_2^p > 0$ for every non-zero \mathbf{z} , then from Theorem 1, \mathbf{x} is the unique solution to the ℓ_p -minimization problem (3). ■

We remark here that ρ^* is a sharp bound for successful recovery. For any $\rho > \rho^*$, from Lemma 2, with overwhelming probability the sum of the largest $\lceil \rho n \rceil$ terms of $|B_i \mathbf{z}|^p$'s is more than the half of the total sum S_1 , i.e. the null space condition stated in Theorem 1 for successful recovery via ℓ_p -minimization fails with overwhelming probability. Therefore, ℓ_p -minimization fails to recover some ρn -sparse vector with overwhelming probability. Proposition 1 implies that the threshold strictly decreases as p increases.

The performance of ℓ_{p_1} -minimization is better than that of ℓ_{p_2} -minimization for $0 < p_1 < p_2 \leq 1$ as ℓ_{p_1} -minimization can recover vectors up to a higher sparsity.

B. Weak Recovery

We have demonstrated in Section III-A that the threshold for strong recovery strictly decreases as p increases from 0 to 1. Here we provide a weak recovery threshold for all $p \in [0, 1)$ when $\alpha \rightarrow 1$. As we shall see, for weak recovery, the threshold of ℓ_p -minimization is the same for all $p \in [0, 1)$, and is lower than the threshold of ℓ_1 -minimization.

Recall that for successful weak recovery, ℓ_p -minimization should recover all the vectors on some fixed support with a fixed sign pattern, and the equivalent null space characterization is stated in Theorem 3 and Theorem 4.

We define $x^0 = 1$ for all $x \neq 0$, and $0^0 = 0$. To characterize the recovery threshold of ℓ_p -minimization in this case, we first state the following lemma,

Lemma 4. *Let X_1, X_2, \dots, X_n be i.i.d. $\mathcal{N}(0, 1)$ random variables and T be a set of indices with size $|T| = \rho n$ for some $\rho > 0$. Let $\mathbf{x} \in \mathcal{R}^n$ be any vector on support T with fixed sign pattern. For every $p \in [0, 1)$, for every $\epsilon > 0$, when n is large enough, with probability at least $1 - e^{-c_6 n}$ for some constant $c_6 > 0$, the following two properties hold simultaneously:*

- $\frac{1}{2}\rho n(\mu - \epsilon) < \sum_{i \in T: X_i x_i < 0} |X_i|^p < \frac{1}{2}\rho n(\mu + \epsilon)$
- $(1 - \rho)n(\mu - \epsilon) < \sum_{i \in T^c} |X_i|^p < (1 - \rho)n(\mu + \epsilon)$.

where $\mu = E[|X|^p]$, $X \sim \mathcal{N}(0, 1)$.

Proof: Define a random variable s_i for each i in T that is equal to 1 if $X_i x_i < 0$ and equal to 0 otherwise. Then $\sum_{i \in T: X_i x_i < 0} |X_i|^p = \sum_{i \in T} |X_i|^p s_i$. $E[|X_i|^p s_i] = \frac{1}{2}\mu$ for every i in T as $X_i \sim \mathcal{N}(0, 1)$. From the Chernoff bound, for any $\epsilon > 0$, there exist $d_1 > 0$ and $d_2 > 0$ such that

$$P\left[\sum_{i \in T} |X_i|^p s_i \leq \frac{1}{2}\rho n(\mu - \epsilon)\right] \leq e^{-d_1 n},$$

$$P\left[\sum_{i \in T} |X_i|^p s_i \geq \frac{1}{2}\rho n(\mu + \epsilon)\right] \leq e^{-d_2 n}.$$

Again from the Chernoff bound, there exist some constants $d_3 > 0$, $d_4 > 0$ such that

$$P\left[\sum_{i \in T^c} |X_i|^p \leq (1 - \rho)n(\mu - \epsilon)\right] \leq e^{-d_3 n},$$

$$P\left[\sum_{i \in T^c} |X_i|^p \geq (1 - \rho)n(\mu + \epsilon)\right] \leq e^{-d_4 n}.$$

By union bound, there exists some constant $c_6 > 0$ such that the two properties stated in the lemma hold at the same time with probability at least $1 - e^{-c_6 n}$.

■

Lemma 4 implies that $\sum_{i \in T: X_i x_i < 0} |X_i|^p < \sum_{i \in T^c} |X_i|^p$ holds with high probability when $|T| = \rho n < \frac{2}{3}n$. Applying the similar net argument in Section III-A, we can extend the result to every vector $B\mathbf{z}$ where matrix $B^{n \times (n-m)}$ has i.i.d. Gaussian entries and \mathbf{z} is any non-zero vector in \mathcal{R}^{n-m} . Then we can establish the main result regarding the threshold of successful recovery with ℓ_p -minimization from vectors on one support with the same sign pattern.

Theorem 6. *For any $p \in [0, 1)$, given any $\rho < \rho_w^* := \frac{2}{3}$, there exist constants $c_7 \in (0, 1)$, $c_8 > 0$ such that when $\alpha > c_7$ and n is large enough, with probability at least $1 - e^{-c_8 n}$, an $m \times n$ matrix A with i.i.d. $\mathcal{N}(0, 1)$ entries has the following property: for every vector \mathbf{x} on some support T satisfying $|T| \leq \rho m$ with fixed sign pattern on T , \mathbf{x} is the unique solution to the ℓ_p -minimization problem.*

Proof: From Lemma 4, applying similar arguments in the proof of Lemma 3, we get that when $\alpha > c_7$ for some $0 < c_7 < 1$ and n is large enough, with probability $1 - e^{-c_8 n}$ for some $c_8 > 0$,

- $\frac{1}{2}\rho n(\mu - \epsilon) < \sum_{i \in T: (B_i \mathbf{v})x_i < 0} |B_i \mathbf{v}|^p < \frac{1}{2}\rho n(\mu + \epsilon)$
- $(1 - \rho)n(\mu - \epsilon) < \sum_{i \in T^c} |B_i \mathbf{v}|^p < (1 - \rho)n(\mu + \epsilon)$

hold for all the vectors \mathbf{v} in a γ -net Σ at the same time. Let \mathcal{S} be the unit sphere in \mathcal{R}^{n-m} . Pick any $\mathbf{z} \in \mathcal{S}$, from (25) we have $\mathbf{z} = \sum_{j \geq 0} \gamma_j \mathbf{v}_j$, where $\gamma_0 = 1$, $\mathbf{v}_j \in \Sigma$ for all j and $\gamma_j \leq \gamma^j$.

Given \mathbf{z} , let $T^- = \{i \in T : B_i \mathbf{z} x_i < 0\}$. For any i in T^- ,

$$\begin{aligned}
 |B_i \mathbf{z}|^p &= \left| \sum_{j \geq 0} \gamma_j B_i \mathbf{v}_j \right|^p \\
 &= \left| \sum_{j: (B_i \mathbf{v}_j)x_i < 0} \gamma_j B_i \mathbf{v}_j + \sum_{j: (B_i \mathbf{v}_j)x_i \geq 0} \gamma_j B_i \mathbf{v}_j \right|^p \\
 &\leq \left| \sum_{j: (B_i \mathbf{v}_j)x_i < 0} \gamma_j B_i \mathbf{v}_j \right|^p \\
 &\leq \sum_{j: (B_i \mathbf{v}_j)x_i < 0} \gamma_j^p |B_i \mathbf{v}_j|^p
 \end{aligned}$$

where the first inequality holds as $(B_i \mathbf{z})_{x_i} < 0$. Then

$$\begin{aligned} \|B_{T^c} \mathbf{z}\|_p^p &\leq \sum_{i \in T^c} \sum_{j: (B_i \mathbf{v}_j)_{x_i} < 0} \gamma^{jp} |B_i \mathbf{v}_j|^p \\ &\leq \sum_{i \in T} \sum_{j: (B_i \mathbf{v}_j)_{x_i} < 0} \gamma^{jp} |B_i \mathbf{v}_j|^p \\ &= \sum_{j \geq 0} \gamma^{jp} \sum_{i \in T: (B_i \mathbf{v}_j)_{x_i} < 0} |B_i \mathbf{v}_j|^p \end{aligned} \quad (26)$$

$$< \frac{1}{2(1 - \gamma^p)} \rho n (\mu + \epsilon). \quad (27)$$

We also have

$$\begin{aligned} \|B_{T^c} \mathbf{z}\|_p^p &= \left\| \left(\sum_{j \geq 0} \gamma_j B_{T^c} \mathbf{v}_j \right) \right\|_p^p \\ &\geq \|B_{T^c} \mathbf{v}_0\|_p^p - \sum_{j \geq 1} \gamma^{jp} \|B_{T^c} \mathbf{v}_j\|_p^p \\ &> (1 - \rho)n(\mu - \epsilon) - \sum_{j \geq 1} \gamma^{jp} (1 - \rho)n(\mu + \epsilon) \\ &\geq (1 - \rho)n \frac{\mu - 2\mu\gamma^p - \epsilon}{1 - \gamma^p}. \end{aligned} \quad (28)$$

Combining (27) and (28), we have for every $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c} \mathbf{z}\|_p^p - \|B_{T^c} \mathbf{z}\|_p^p > \frac{n\mu}{1 - \gamma^p} (1 - \frac{3}{2}\rho - 2\gamma^p(1 - \rho) - \frac{\epsilon}{\mu}(1 - \frac{\rho}{2}))$. Then for every non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, we have $\|B_{T^c} \mathbf{z}\|_p^p - \|B_{T^c} \mathbf{z}\|_p^p > \|\mathbf{z}\|_2^p \frac{n\mu}{1 - \gamma^p} (1 - \frac{3}{2}\rho - 2\gamma^p(1 - \rho) - \frac{\epsilon}{\mu}(1 - \frac{\rho}{2}))$. For any $\rho < \frac{2}{3}$, we can pick γ and ϵ small enough such that the righthand side is positive. The result follows by applying Theorem 3 and Theorem 4. ■

We remark here that ρ_w^* is a sharp bound for successful recovery in this setup. For any $\rho > \rho_w^*$, from Lemma 4, with overwhelming probability that $\sum_{i \in T: X_i h_i < 0} |X_i|^p > \sum_{i \in T^c} |X_i|^p$, then Theorem 3 and Theorem 4 indicate that the ℓ_p -minimization ($p \in [0, 1)$) fails to recover some ρn -sparse vector \mathbf{x} in this case. Note that for a random Gaussian measurement matrix, from symmetry one can check that this results does not depend on the specific choice of support and sign pattern. In fact, Theorem 6 holds for any fixed support and any fixed sign pattern.

Surprisingly, the successful recovery threshold ρ_w^* when we only consider recovering vectors on one support with one sign pattern is $\frac{2}{3}$ for all p in $[0, 1)$ and is strictly less than the threshold for $p = 1$, which is 1 ([15]). Thus in this case, ℓ_1 -minimization has better recovery performance than ℓ_p -minimization ($p \in [0, 1)$) in terms of the sparsity requirement for the sparse vector. If we view the ability to recover all the vectors up to certain sparsity as the “worst” case performance, and the ability to recovery all the

sparse vectors on one support with one sign pattern as the “expected” case performance, then although worst case performance can be improved if we apply ℓ_p -minimization with a smaller p , ℓ_1 -minimization in fact has the best expected case performance for all $p \in [0, 1]$.

It might be counterintuitive at first sight to see that the weak threshold of ℓ_0 -minimization is less than that of ℓ_1 -minimization, so let us take a moment to consider what the result means. We choose recovering all nonnegative vectors on some support T ($|T| = \rho n$) for the weak recovery, the argument follows for all the other supports and all the other sign patterns. The results about weak recovery threshold indicate that for any $\rho \in (2/3, 1)$, when n is sufficiently large and $\alpha \rightarrow 1$, for a random Gaussian measurement matrix A , ℓ_1 -minimization would recover all the nonnegative vectors on some support T ($|T| = \rho n$) with overwhelming probability, while ℓ_0 -minimization would fail to recover some nonnegative vector on T with overwhelming probability according to Theorem 6. This can happen when there exists a nonnegative vector \mathbf{x} on support T and a vector \mathbf{x}' on support T' such that $|T'| \leq |T|$, and $A\mathbf{x} = A\mathbf{x}'$. Note that \mathbf{x}' could have negative entries, or T' may not be a subset of T . Therefore, if \mathbf{x} is the sparse vector we would like to recover from $A\mathbf{x}$, ℓ_0 -minimization would fail since $\|\mathbf{x}'\|_0 \leq \|\mathbf{x}\|_0$. However, $\|\mathbf{x}\|_1 < \|\mathbf{x}'\|_1$ should hold since ℓ_1 -minimization can successfully return \mathbf{x} as its solution. Of course when \mathbf{x}' is the sparse vector we would like to recover, ℓ_1 -minimization would return \mathbf{x} and fail to recover \mathbf{x}' . However, since ℓ_1 -minimization would recover all the nonnegative vectors on T , then either $T' \not\subseteq T$ holds or \mathbf{x}' has negative entries. Therefore when we consider recovering nonnegative vectors on T for the weak recovery, \mathbf{x}' is not taken into account, and ℓ_1 -minimization works better than ℓ_0 -minimization. Therefore, although the performance of ℓ_1 -minimization is not as good as that of ℓ_p -minimization ($p \in [0, 1)$) in the strong recovery which requires to recover all the vectors up to certain sparsity, ℓ_1 -minimization can recover all the ρn -sparse ($\rho > 2/3$) vectors on some support with some sign pattern, while for ℓ_p -minimization ($p \in [0, 1)$), the size of the largest support on which it can recover all the vectors with one sign pattern is no greater than $2n/3$. Thus, when we aim to recover all the vectors up to certain sparsity, ℓ_p -minimization is better for smaller p , however, when we aim to recover all the vectors on one support with one sign pattern, ℓ_1 -minimization may have a better performance.

IV. RECOVERY BOUNDS FOR EVERY $\lim_{n \rightarrow \infty} \frac{m}{n} < 1$

We considered the limiting case that $\alpha \rightarrow 1$ in Section III and provided the limiting thresholds of sparsity ratio for successful recovery via ℓ_p -minimization both in the strong sense and in the weak sense. Here we focus on the case that α is given ($0 < \alpha < 1$). For any α and p , we will provide a bound $\rho^*(\alpha, p)$ for strong recovery and a bound $\rho_w^*(\alpha, p)$ for weak recovery such that ℓ_p -minimization can recover all the

$\rho^*(\alpha, p)n$ -sparse vectors with overwhelming probability, and recover all the $\rho_w^*(\alpha, p)n$ -sparse vectors on one support with one sign pattern with overwhelming probability. Note that the thresholds we provided in Section III is tight in the sense that for any $\rho > \rho^*$ in the strong recovery or any $\rho > \rho_w^*$ in the weak recovery, with overwhelming probability ℓ_p -minimization would fail to recover some ρn sparse vector. However, $\rho^*(\alpha, p)$ and $\rho_w^*(\alpha, p)$ we provide in this section are lower bounds for the thresholds of strong recovery and weak recovery respectively, and might not be tight in general.

A. Strong Recovery

As discussed in Section III, since A has i.i.d. $\mathcal{N}(0, 1)$ entries, there exists a basis B of the null space of A with i.i.d. $\mathcal{N}(0, 1)$ entries. Let \mathcal{S} be the unit sphere in \mathcal{R}^{n-m} . From Theorem 1 we know that in order to successfully recover all the ρn -sparse vectors via ℓ_p -minimization, $\|B_T \mathbf{z}\|_p^p < \frac{1}{2} \|B \mathbf{z}\|_p^p$ should hold for every non-zero vector $\mathbf{z} \in \mathcal{R}^n$, and every set $T \subset \{1, \dots, n\}$ with $|T| \leq \rho n$. We will first establish a lower bound of $\|B \mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ with overwhelming probability in Lemma 5. Lemma 6 establishes the fact that for any given constant $c > 0$, there always exists some $\rho > 0$ such that $\|B_T \mathbf{z}\|_p^p \leq cn$ for all $\mathbf{z} \in \mathcal{S}$ and all T with $|T| \leq \rho n$ with overwhelming probability. Combining Lemma 5 and Lemma 6 we will establish a positive lower bound $\rho^*(\alpha, p)$ of sparsity ratio for successful recovery for every $\alpha \in (0, 1)$ and every $p \in (0, 1]$ in Theorem 7.

Lemma 5. *For any α and p , there exists a constant $\lambda_{\min}(\alpha, p) > 0$ and some constant $c_9 > 0$ such that with probability at least $1 - e^{-c_9 n}$, for every $\mathbf{z} \in \mathcal{S}$, $\|B \mathbf{z}\|_p^p > \lambda_{\min}(\alpha, p)n$.*

Lemma 6. *Given any α, p and corresponding $\lambda_{\min}(\alpha, p) > 0$, there exists a constant $\rho^*(\alpha, p) > 0$ and some constant $c_{10} > 0$ such that with probability at least $1 - e^{-c_{10} n}$, for every $\mathbf{z} \in \mathcal{S}$ and for every set $T \subset \{1, 2, \dots, m\}$ with $|T| \leq \rho^*(\alpha, p)m$, $\|B_T \mathbf{z}\|_p^p < \frac{1}{2} \lambda_{\min}(\alpha, p)n$.*

We defer the proofs of Lemma 5 and Lemma 6 for later discussion, and first present our result on bounds for strong recovery of ℓ_p -minimization with given $\alpha \in (0, 1)$.

Theorem 7. *For any $0 < p \leq 1$, for matrix $A^{m \times n}$ ($\alpha = \frac{m}{n}$) with i.i.d $\mathcal{N}(0, 1)$ entries, there exists a constant $c_{11} > 0$ such that with probability at least $1 - e^{-c_{11} n}$, \mathbf{x} is the unique solution to the ℓ_p -minimization problem (3) for every vector \mathbf{x} up to $\rho^*(\alpha, p)n$ -sparse.*

Proof: Let \mathcal{S} be the unit sphere in \mathcal{R}^{n-m} . Then

$$\begin{aligned}
& P(\text{Strong recovery succeeds to recover vectors up to } \rho^*(\alpha, p)n\text{-sparse}) \\
&= P(\forall \text{ non-zero } \mathbf{z} \in \mathcal{R}^{n-m}, \forall T \text{ with } |T| = \rho^*(\alpha, p)n, \|B_T \mathbf{z}\|_p^p < \frac{1}{2} \|B \mathbf{z}\|_p^p) \\
&= P(\forall \mathbf{z} \in \mathcal{S}, \forall T \text{ with } |T| = \rho^*(\alpha, p)n, \|B_T \mathbf{z}\|_p^p < \frac{1}{2} \|B \mathbf{z}\|_p^p) \\
&\geq P(\forall \mathbf{z} \in \mathcal{S}, \forall T \text{ with } |T| = \rho^*(\alpha, p)n, \|B_T \mathbf{z}\|_p^p < \frac{1}{2} \lambda_{\min}(\alpha, p)n, \text{ and } \|B \mathbf{z}\|_p^p > \lambda_{\min}(\alpha, p)n) \\
&\geq 1 - P(\exists \mathbf{z} \in \mathcal{S}, \text{ s.t. } \|B \mathbf{z}\|_p^p \leq \lambda_{\min}(\alpha, p)n) \\
&\quad - P(\exists \mathbf{z} \in \mathcal{S}, \exists T \text{ with } |T| = \rho^*(\alpha, p)n \text{ s.t. } \|B_T \mathbf{z}\|_p^p \geq \lambda_{\min}(\alpha, p)n/2) \\
&= 1 - e^{-c_9 n} - e^{-c_{10} n}, \tag{29}
\end{aligned}$$

where the first equality follows from Theorem 1, the second equality holds since for any non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, $\mathbf{z}/\|\mathbf{z}\|_2 \in \mathcal{S}$. From Lemma 5 we know there exists $c_9 > 0$ such that $P(\exists \mathbf{z} \in \mathcal{S}, \text{ s.t. } \|B \mathbf{z}\|_p^p \leq \lambda_{\min}(\alpha, p)n) \leq e^{-c_9 n}$, and from Lemma 6 we know there exists $c_{10} > 0$ such that $P(\exists \mathbf{z} \in \mathcal{S}, \exists T \text{ s.t. } \|B_T \mathbf{z}\|_p^p \geq \frac{1}{2} \lambda_{\min}(\alpha, p)n) \leq e^{-c_{10} n}$, then there exists $c_{11} > 0$ which depends on α, p and λ_{\min} such that (29) $\geq 1 - e^{-c_{11} n}$. Therefore, ℓ_p -minimization can recover all the $\rho^*(\alpha, p)n$ -sparse vectors with probability at least $1 - e^{-c_{11} n}$. ■

Theorems 7 implies that for every $\alpha \in (0, 1)$ and every $p \in (0, 1]$, there exists a positive constant $\rho^*(\alpha, p)$ such that ℓ_p -minimization can recover all the ρ^*n -sparse vectors with overwhelming probability. Since $\rho^*(\alpha, p)$ is a lower bound of the threshold of the strong recovery, we want it to be as high as possible. Next we show how to calculate $\rho^*(\alpha, p)$ and improve it as much as possible. In order to calculate $\rho^*(\alpha, p)$, we first calculate $\lambda_{\min}(\alpha, p)$ in Lemma 5, and then with the obtained $\lambda_{\min}(\alpha, p)$, we can calculate $\rho^*(\alpha, p)$ in Lemma 6. We want to obtain $\lambda_{\min}(\alpha, p)$ which is as large as possible while Lemma 5 still holds, and given $\lambda_{\min}(\alpha, p)$, we want $\rho^*(\alpha, p)$ to be as large as possible while Lemma 6 still holds. How to calculate $\lambda_{\min}(\alpha, p)$ and $\rho^*(\alpha, p)$ is stated in the following text, and Lemma 5 and Lemma 6 are proved in the meantime. The values of $\lambda_{\min}(\alpha, p)$ and $\rho^*(\alpha, p)$ can be computed from (38) and (43).

1) Calculation of $\lambda_{\min}(\alpha, p)$ in Lemma 5:

Given α and p , define

$$c_{\max} = \frac{1}{n} \sup_{\mathbf{z} \in \mathcal{S}} \|B \mathbf{z}\|_p^p = \frac{1}{n} \max_{\mathbf{z} \in \mathcal{S}} \|B \mathbf{z}\|_p^p,$$

where the second equality holds by compactness. Thus, for any non-zero vector \mathbf{z} , $\|B \mathbf{z}\|_p^p \leq \|\mathbf{z}\|_p^p c_{\max} n$.

Define

$$c_{\min} = \frac{1}{n} \min_{\mathbf{z} \in \mathcal{S}} \|B\mathbf{z}\|_p^p.$$

Pick a γ -net Σ_2 of \mathcal{S} with cardinality at most $(1 + 2/\gamma)^{n-m}$ [28] and $\gamma > 0$ to be chosen later, we define

$$\theta = \frac{1}{n} \min_{\mathbf{z} \in \Sigma_2} \|B\mathbf{z}\|_p^p.$$

Then for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}' \in \Sigma_2$ such that $\|\mathbf{z} - \mathbf{z}'\|_2 \leq \gamma$. We have

$$\|B\mathbf{z}\|_p^p \geq \|B\mathbf{z}'\|_p^p - \|B(\mathbf{z} - \mathbf{z}')\|_p^p \geq \theta n - \gamma^p c_{\max} n, \quad (30)$$

where the first inequality follows from triangular inequality and the second inequality follows from the definition of c_{\max} . Since (30) holds for every \mathbf{z} in \mathcal{S} , we have

$$c_{\min} \geq \theta - \gamma^p c_{\max}. \quad (31)$$

To calculate $\lambda_{\min}(\alpha, p)$, we essentially need to characterize c_{\min} . From (31), we can achieve this by characterizing θ and c_{\max} .

We first show that there exists constant $b > 0$ such that with overwhelming probability, $\theta > b$ holds, i.e. $\|B\mathbf{z}\|_p^p > bn$ for all \mathbf{z} in Σ_2 .

$$\begin{aligned} P(\theta \leq b) &= P(\exists \mathbf{z} \in \Sigma_2 \text{ s.t. } \|B\mathbf{z}\|_p^p \leq bn) \\ &\leq \sum_{\mathbf{z} \in \Sigma_2} P(\|B\mathbf{z}\|_p^p \leq bn) \\ &\leq (1 + 2/\gamma)^{n-m} e^{tbn} E[e^{-t \sum_i |B_i \mathbf{z}|^p}], \quad \forall t > 0 \\ &= (1 + 2/\gamma)^{(1-\alpha)n} e^{tbn} E[e^{-t|X|^p}]^n, \quad \forall t > 0 \\ &= e^{((1-\alpha) \log(1+2/\gamma) + \log(E[e^{-t|X|^p}]) + bt)n}, \quad \forall t > 0, \end{aligned} \quad (32)$$

where $X \sim \mathcal{N}(0, 1)$. The first inequality follows from the union bound and the fact that $P(\|B\mathbf{z}\|_p^p \leq bn)$ is the same for all $\mathbf{z} \in \Sigma_2$ since B has i.i.d. $\mathcal{N}(0, 1)$ entries. The second inequality follows from the Chernoff bound. Note that

$$\begin{aligned} E[e^{-t|X|^p}] &= \sqrt{2/\pi} \int_0^\infty e^{-tx^p} e^{-\frac{1}{2}x^2} dx \\ &= t^{-\frac{1}{p}} \sqrt{2/\pi} \int_0^\infty e^{-y^p} e^{-\frac{1}{2}(t^{-\frac{1}{p}}y)^2} dy. \end{aligned} \quad (33)$$

$$\begin{aligned} &\leq t^{-\frac{1}{p}} \sqrt{2/\pi} \int_0^\infty e^{-y^p} dy \\ &= t^{-\frac{1}{p}} \sqrt{2/\pi} \Gamma(1/p)/p, \end{aligned} \quad (34)$$

where (33) holds from changing variables using $x = t^{-\frac{1}{p}}y$, and the inequality follows from the fact that $e^{-\frac{1}{2}(t^{-\frac{1}{p}}y)^2} \leq 1$ for all $y \geq 0$. If it further holds that $t > 1$, then $t^{-\frac{1}{p}} < 1$. Then from (33) we have

$$E[e^{-t|X|^p}] \geq t^{-\frac{1}{p}} \sqrt{2/\pi} \int_0^\infty e^{-y^p - \frac{1}{2}y^2} dy.$$

Since $\int_0^\infty e^{-y^p - \frac{1}{2}y^2} dy$ exists and is positive, then combining (34) and (35), we have

$$E[e^{-t|X|^p}] = O(t^{-\frac{1}{p}}). \quad (35)$$

Since (32) holds for all $t > 0$, we let $t = \gamma^{-p(1-\alpha+\epsilon)}$ for any ϵ such that $0 < \epsilon \leq \alpha$ and let $b(\gamma) = 1/t$, then from (32) we have

$$P(\theta \leq b(\gamma)) \leq e^{((1-\alpha)\log(1+2/\gamma) + \log(O(\gamma^{1-\alpha+\epsilon})) + 1)n} = e^{-\kappa n},$$

where $\kappa(\gamma) = -(1-\alpha)\log(1 + \frac{2}{\gamma}) - \log(O(\gamma^{1-\alpha+\epsilon})) - 1$. Note that since $\epsilon > 0$, when γ is sufficiently small, $\kappa(\gamma) > 0$. Therefore when $\gamma \leq \xi$ for some small $\xi > 0$, there exists constant $\kappa(\gamma) > 0$ such that

$$P(\theta \leq b(\gamma) = \gamma^{p(1-\alpha+\epsilon)}) \leq e^{-\kappa(\gamma)n}. \quad (36)$$

We next show that there exists some $\lambda_{\max}(\alpha, p) > 0$ such that with overwhelming probability, $c_{\max} < \lambda_{\max}(\alpha, p)$ holds. In fact, we have the following Lemma:

Lemma 7. *Given any α and p , there exists a constant $\lambda_{\max}(\alpha, p) > 0$ and some constant $c_{12} > 0$ such that with probability at least $1 - e^{-c_{12}n}$, for every $\mathbf{z} \in \mathcal{S}$, $\|B\mathbf{z}\|_p^p < \lambda_{\max}(\alpha, p)n$.*

Lemma 7 indicates that there exists $\lambda_{\max}(\alpha, p)$ and $c_{12} > 0$ such that

$$P(c_{\max} < \lambda_{\max}(\alpha, p)) \geq 1 - e^{-c_{12}n}. \quad (37)$$

Please refer to the Appendix for the calculation of $\lambda_{\max}(\alpha, p)$, and Lemma 7 is proved in the meantime. In order to obtain a good bound of recovery threshold, we want $\lambda_{\max}(\alpha, p)$ to be as small as possible while Lemma 7 still holds. The numerical value of $\lambda_{\max}(\alpha, p)$ can be computed from (50).

Then after characterizing θ and c_{\max} separately, we are ready to characterize c_{\min} .

$$\begin{aligned} & P(c_{\min} \leq \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p)) \\ & \leq P(\theta - \gamma^p c_{\max} \leq \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p)) \\ & \leq P(\theta \leq \gamma^{p(1-\alpha+\epsilon)}) + P(c_{\max} \geq \lambda_{\max}(\alpha, p)) \\ & \leq e^{-\kappa n} + e^{-c_{12}n}, \end{aligned}$$

where the first inequality follows from (31), and the last inequality follows from (36) and (37). Then for any $\gamma \leq \xi$, there exists constant $c_9 > 0$ such that $P(c_{\min} \leq \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p)) \leq e^{-c_9 n}$. Given $\lambda_{\max}(\alpha, p)$, let

$$\lambda_{\min}(\alpha, p) = \max_{0 < \gamma \leq \xi} \gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max}(\alpha, p). \quad (38)$$

Note that since $1 - \alpha + \epsilon < 1$, $\gamma^{p(1-\alpha+\epsilon)} - \gamma^p \lambda_{\max} > 0$ when γ is sufficiently small, therefore $\lambda_{\min} > 0$, and Lemma 5 follows.

2) *Calculation of $\rho^*(\alpha, p)$ in Lemma 6:*

For any given set $T \subset \{1, 2, \dots, n\}$ with $|T| = \rho n$ ($0 < \rho < 1$), define

$$d_{\max} = \frac{1}{n} \max_{\mathbf{z} \in \mathcal{S}} \|B_T \mathbf{z}\|_p^p.$$

Given a γ -net Σ_3 of \mathcal{S} with cardinality at most $(1 + 2/\gamma)^{n-m}$ and $\gamma > 0$ to be chosen later, define

$$\tau = \frac{1}{n} \max_{\mathbf{z} \in \Sigma_3} \|B_T \mathbf{z}\|_p^p.$$

Then for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}' \in \Sigma_3$ such that $\|\mathbf{z} - \mathbf{z}'\|_2 \leq \gamma$. Then for every $\mathbf{z} \in \mathcal{S}$, we have $\|B_T \mathbf{z}\|_p^p \leq \|B_T \mathbf{z}'\|_p^p + \|B_T(\mathbf{z} - \mathbf{z}')\|_p^p \leq \tau n + \gamma^p d_{\max} n$. Thus,

$$d_{\max} \leq \tau / (1 - \gamma^p). \quad (39)$$

Given $\lambda_{\min}(\alpha, p)$ (denoted by λ_{\min} here for simplicity), in order to obtain $\rho^*(\alpha, p)$ such that Lemma 6 holds, we essentially need to find ρ such that for any T with its corresponding d_{\max} , with overwhelming probability $d_{\max} < \lambda_{\min}/2$ holds for all T with $|T| = \rho m$ at the same time. From (39), we first consider the probability that $\tau \geq \lambda_{\min}(1 - \gamma^p)/2$ holds for a given set T .

$$\begin{aligned} & P(\tau \geq \lambda_{\min}(1 - \gamma^p)/2, \text{ given } T) \\ &= P(\exists \mathbf{z} \in \Sigma_3 \text{ s.t. } \|B_T \mathbf{z}\|_p^p \geq \lambda_{\min}(1 - \gamma^p)n/2) \\ &\leq \sum_{\mathbf{z} \in \Sigma_3} P(\|B_T \mathbf{z}\|_p^p \geq \frac{\lambda_{\min}(1 - \gamma^p)n}{2}) \\ &= \sum_{\mathbf{z} \in \Sigma_3} P(\sum_{i \in T} |B_i \mathbf{z}|^p \geq \frac{\lambda_{\min}(1 - \gamma^p)n}{2}) \\ &\leq (1 + 2/\gamma)^{n-m} \min_{t > 0} e^{-t \lambda_{\min}(1 - \gamma^p)n/2} E[e^{t \sum_{i \in T} |B_i \mathbf{z}|^p}] \\ &= (1 + 2/\gamma)^{(1-\alpha)n} \min_{t > 0} e^{-t \lambda_{\min}(1 - \gamma^p)n/2} E[e^{t |X|^p}]^{\rho n} \\ &= e^{((1-\alpha) \log(1 + \frac{2}{\gamma}) + \min_{t > 0} (\rho \log(E[e^{t |X|^p}]) - t \lambda_{\min}(1 - \gamma^p)/2))n}, \end{aligned} \quad (40)$$

where $X \sim \mathcal{N}(0, 1)$, the first inequality follows from the union bound and the fact that the second inequality follows from the Chernoff bound. Note that since B has i.i.d. $\mathcal{N}(0, 1)$ entries, (40) holds for any T as long as $|T| = \rho n$.

Given ρ , λ_{\min} and γ , since the second derivative of $\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1 - \gamma^p)/2$ to t is positive, then its minimum is achieved where its first derivative is 0.

$$\begin{aligned}
0 &= \frac{d[\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1 - \gamma^p)/2]}{dt} \\
&= \frac{d}{dt}(\rho \log(\sqrt{\frac{2}{\pi}} \int_0^\infty e^{tx^p - \frac{1}{2}x^2} dx) - t\lambda_{\min}(1 - \gamma^p)/2) \\
&= \frac{\rho \int_0^\infty x^p e^{tx^p - \frac{1}{2}x^2} dx}{\int_0^\infty e^{tx^p - \frac{1}{2}x^2} dx} - \lambda_{\min}(1 - \gamma^p)/2.
\end{aligned} \tag{41}$$

Note that when $\rho < \lambda_{\min}(1 - \gamma^p)/(2E[|X|^p])$, the solution of t to (41) is always positive, thus it is also the solution to $\min_{t>0}(\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1 - \gamma^p)/2)$. Now consider the probability that $\|B_T \mathbf{z}\|^p \geq \frac{1}{2}\lambda_{\min}n$ for some $\mathbf{z} \in \mathcal{S}$ and T with $|T| = \rho n$.

$$\begin{aligned}
&P(\exists \mathbf{z} \in \mathcal{S}, \exists T \text{ s.t. } |T| = \rho n, \|B_T \mathbf{z}\|_p^p \geq \lambda_{\min}n/2) \\
&\leq \binom{n}{\rho n} P(\exists \mathbf{z} \in \mathcal{S} \text{ s.t. } \|B_T \mathbf{z}\|_p^p \geq \lambda_{\min}n/2, \\
&\quad \text{for given } T \subset \{1, 2, \dots, n\} \text{ and } |T| = \rho n) \\
&= \binom{n}{\rho n} P(d_{\max} \geq \lambda_{\min}/2) \\
&\leq \binom{n}{\rho n} P(\tau/(1 - \gamma^p) \geq \lambda_{\min}/2) \\
&= \binom{n}{\rho n} P(\tau \geq \lambda_{\min}(1 - \gamma^p)/2) \\
&\leq 2^{nH(\rho)} e^{((1-\alpha)\log(1+2/\gamma) + \min_{t>0}(\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1 - \gamma^p)/2))n} \\
&= e^{(H(\rho)\log 2 + (1-\alpha)\log(1+2/\gamma) + \min_{t>0}(\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1 - \gamma^p)/2))n},
\end{aligned} \tag{42}$$

where the first inequality follows from the union bound and the second inequality follows from (39). Note that given α , p , and λ_{\min} , for every γ , as $\rho \rightarrow 0$, $H(\rho)$ goes to 0, and $\min_{t>0}(\rho \log(E[e^{t|X|^p}]) - t\lambda_{\min}(1 - \gamma^p)/2)$ goes to $-\infty$, thus, there exists $\rho(\alpha, p, \gamma) > 0$ such that the exponent of (42) is negative for all $\rho \leq \rho(\alpha, p, \gamma)$. In other words, for each γ , there exists some $c_{10} > 0$ such that (42) $\leq e^{-c_{10}n}$ when $\rho = \rho(\alpha, p, \gamma)$. Then, with probability at least $1 - e^{-c_{10}n}$, for every $\mathbf{z} \in \mathcal{S}$ and for every set $T \subset \{1, 2, \dots, n\}$ with $|T| \leq \rho(\gamma)n$, $\|B_T \mathbf{z}\|_p^p < \lambda_{\min}n/2$. Let

$$\rho^*(\alpha, p) = \max_{\gamma} \rho(\alpha, p, \gamma), \tag{43}$$

then Lemma 6 follows.

Theorem 7 establishes the existence of $\rho^*(\alpha, p) > 0$ for all $0 < \alpha < 1$ and $0 < p \leq 1$ such that ℓ_p -minimization can recover all the $\rho^*(\alpha, p)n$ -sparse vectors with overwhelming probability. We numerically calculate this bound by calculating first $\lambda_{\max}(\alpha, p)$ in Lemma 7 from (50), and then $\lambda_{\min}(\alpha, p)$ in Lemma 5 from (38), and finally $\rho^*(\alpha, p)$ in Lemma 6 from (43). Fig. 2 shows the curve of $\rho^*(\alpha, p)$ against α for different p , and Fig. 3 shows the curve of $\rho^*(\alpha, p)$ against p for different α . Note that for any p , $\lim_{\alpha \rightarrow 1} \rho^*(\alpha, p)$ is slightly smaller than the limiting threshold of strong recovery we obtained in Section III-A. For example, when $p = 0.5$, the threshold $\rho^*(0.5)$ we obtained in Section III-A is 0.3406, and the bound $\rho^*(\alpha, 0.5)$ we obtained here is approximately 0.268 when α goes to 1. This is because in Section III-A we employed a finer technique to characterize the sum of the largest ρn terms of n i.i.d. random variables directly, while in Section IV-A introducing the union bound causes some slackness.

Compared with the bound obtained in [4] through restricted isometry condition, our bound $\rho^*(\alpha, p)$ is tighter when α is relatively large. For example, when $p = 1$, the bound in [4] (Fig.3.2(a)) is in the order of 10^{-3} for all $\alpha \in (0, 1)$ and upper bounded by 0.0035, while $\rho^*(\alpha, 1)$ is greater than 0.0039 for all $\alpha \geq 0.8$ and increases to 0.1308 as $\alpha \rightarrow 1$. When $p = 0.5$, the bound in [4] (Fig.3.2(c)) is in the order of 10^{-3} for all $\alpha \in (0, 1)$ and upper bounded by 0.01, while here $\rho^*(\alpha, 0.5)$ is greater than 0.011 for all $\alpha \geq 0.65$ and increases to 0.268 as $\alpha \rightarrow 1$. Therefore, although [4] provides a better bound than ours when α is small, our bound ρ^* improves over that in [4] when α is relatively large. [15] applies geometric face counting technique to the strong bound of successful recovery of ℓ_1 -minimization (Fig.1.1). Since if the necessary and sufficient condition (4) is satisfied for $p = 1$, then it is also satisfied for all $p < 1$, therefore the bound in [17] can serve as the bound of successful recovery for all $0 < p < 1$. Our bound $\rho^*(\alpha, p)$ in Section IV is higher than that in [15] when α is relatively large.

B. Weak Recovery

Theorem 3 provides a sufficient condition for successful recovery of every ρn -sparse vector \mathbf{x} on one support T with one sign pattern, which requires $\|B_{T^-} \mathbf{z}\|_p^p < \|B_{T^c} \mathbf{z}\|_p^p$ to hold for all non-zero $\mathbf{z} \in \mathcal{R}^n$, where given \mathbf{z} , $T^- = \{i : B_i \mathbf{z} x_i < 0\}$. Given α, p and $\rho \in (0, 1)$, we will establish a lower bound of $\|B_{T^c} \mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ in Lemma 8, and establish an upper bound of $\|B_{T^-} \mathbf{z}\|_p^p$ in Lemma 9. If there exists $\rho_w^*(\alpha, p) > 0$ such that the corresponding lower bound of $\|B_{T^c} \mathbf{z}\|_p^p$ is greater than the upper bound of $\|B_{T^-} \mathbf{z}\|_p^p$, which in fact is always true as we will see in Theorem 8, then $\rho_w^*(\alpha, p)$ serves as a lower bound of recovery threshold of ℓ_p -minimization for vectors on a fixed support with a fixed sign pattern.

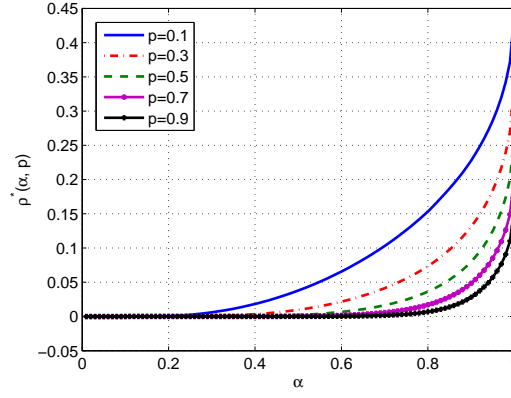


Fig. 2. $\rho^*(\alpha, p)$ against α for different p

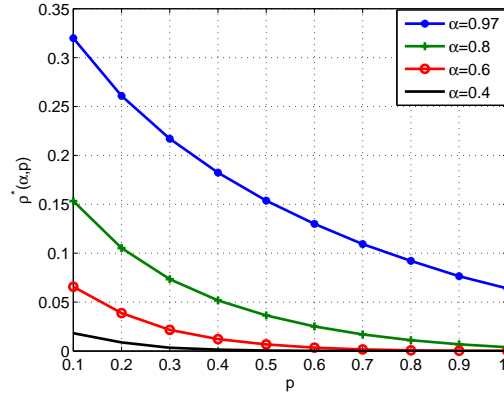


Fig. 3. $\rho^*(\alpha, p)$ against p for different α

The technique to establish the lower bound of $\|B_{T^c}\mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ is the same as that in Lemma 5. We state the result in Lemma 8, please refer to the appendix for its proof.

Lemma 8. *Given α, p and set $T \subset \{1, \dots, n\}$ with $|T| = \rho n$, with probability at least $1 - e^{-c_{13}n}$ for some $c_{13} > 0$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p < (1 - \rho)\lambda_{\max}(\frac{\alpha - \rho}{1 - \rho}, p)n$, and with probability at least $1 - e^{-c_{14}n}$ for some $c_{14} > 0$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c}\mathbf{z}\|_p^p > (1 - \rho)\lambda_{\min}(\frac{\alpha - \rho}{1 - \rho}, p)n$, where $\lambda_{\max}(\alpha, p)$ and $\lambda_{\min}(\alpha, p)$ are defined in (50) and (38) respectively.*

Given T with $|T| = \rho n$, Lemma 8 provides a lower bound of $\|B_{T^c}\mathbf{z}\|_p^p$ which holds with overwhelming probability for all $\mathbf{z} \in \mathcal{S}$. Please refer to the Appendix for its proof. Next we will provide an upper bound of $\|B_{T^c}\mathbf{z}\|_p^p$ for all $\mathbf{z} \in \mathcal{S}$ in Lemma 9. One should be cautious that the set T^- varies for different \mathbf{z} . To

improve the bound of the threshold of successful weak recovery, we want $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ to be as small as possible while Lemma 9 still holds. $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ can be computed from (57), please refer to the Appendix for its detailed calculation.

Lemma 9. *Given α , p and set $T \subset \{1, \dots, n\}$ with $|T| = \rho n$, with probability at least $1 - e^{-c_{15}n}$ for some $c_{15} > 0$, for every $\mathbf{z} \in \mathcal{S}$, $\|B_{T-\mathbf{z}}\|_p^p < \rho \tilde{\lambda}_{\max}(\alpha, p, \rho)n$, for some $\tilde{\lambda}_{\max}(\alpha, p, \rho) > 0$.*

With the help of Lemma 8 and Lemma 9, we are ready to present the result regarding the lower bound of recovery threshold via ℓ_p -minimization in the weak sense for given α .

Theorem 8. *For any $0 < p \leq 1$, for matrix $A^{m \times n}$ with i.i.d $\mathcal{N}(0, 1)$ entries, there exists constant $\rho_w^*(\alpha, p) > 0$ and $c_{16} > 0$ such that with probability at least $1 - e^{-c_{16}n}$, \mathbf{x} is the unique solution to the ℓ_p -minimization problem (3) for every $\rho_w^*(\alpha, p)n$ -sparse vector \mathbf{x} on one support T with one sign pattern.*

Proof: Note that given p and α , since $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ and $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ are both positive for all $\rho \in (0, 1)$, and one can check from the definition of $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ and $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ that when ρ decreases, $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ is non-increasing, and $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ is non-decreasing. Therefore, there always exists $\rho_w^*(\alpha, p) > 0$ (denoted by ρ_w^* for simplicity here) such that

$$\rho_w^* \tilde{\lambda}_{\max}(\alpha, p, \rho_w^*) \leq (1 - \rho_w^*) \lambda_{\min}(\frac{\alpha - \rho_w^*}{1 - \rho_w^*}, p). \quad (44)$$

Now consider the probability that ℓ_p -minimization can recover all the ρ_w^*n -sparse \mathbf{x} on one fixed support T with one fixed sign pattern. From Theorem 3 we know that $\|B_{T-\mathbf{z}}\|_p^p < \|B_{T^c\mathbf{z}}\|_p^p$ for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$ is a sufficient condition for the success of weak recovery, thus

$$\begin{aligned} & P(\text{Weak recovery succeeds up to } \rho_w^*n\text{-sparse}) \\ & \geq P(\forall \text{ non-zero } \mathbf{z} \in \mathcal{R}^{n-m}, \|B_{T-\mathbf{z}}\|_p^p < \|B_{T^c\mathbf{z}}\|_p^p) \\ & = P(\forall \mathbf{z} \in \mathcal{S}, \|B_{T-\mathbf{z}}\|_p^p < \|B_{T^c\mathbf{z}}\|_p^p) \\ & \geq P(\forall \mathbf{z} \in \mathcal{S}, \|B_{T-\mathbf{z}}\|_p^p < \rho_w^* \tilde{\lambda}_{\max}(\alpha, p, \rho_w^*), \text{ and} \\ & \quad \|B_{T^c\mathbf{z}}\|_p^p > (1 - \rho_w^*) \lambda_{\min}(\frac{\alpha - \rho_w^*}{1 - \rho_w^*}, p)) \\ & \geq 1 - e^{-c_{15}n} - e^{-c_{14}n}, \end{aligned} \quad (45)$$

where the equality holds since for any non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$, $\mathbf{z}/\|\mathbf{z}\|_2 \in \mathcal{S}$, and the second inequality follows from (44). From Lemma 8 we know there exists $c_{14} > 0$ such that $P(\|B_{T^c\mathbf{z}}\|_p^p > (1 - \rho_w^*) \lambda_{\min}(1 -$

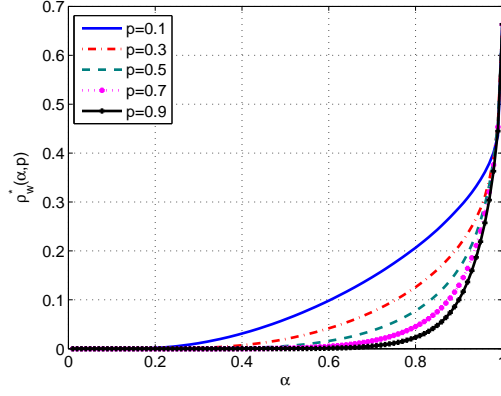


Fig. 4. $\rho_w^*(\alpha, p)$ against α for different p

$\frac{1-\alpha}{1-\rho_w^*}, p)) \geq 1 - e^{-c_{14}n}$, and from Lemma 9 we know there exists $c_{15} > 0$ such that $P(\forall \mathbf{z} \in \mathcal{S}, \|B_T - \mathbf{z}\|_p^p < \rho_w^* \tilde{\lambda}_{\max}(\alpha, p, \rho_w^*)) \geq 1 - e^{-c_{14}n}$, then (45) holds. Thus, there exists $c_{16} > 0$ such that with probability at least $1 - e^{-c_{16}n}$, ℓ_p -minimization problem can recover all $\rho_w^* n$ -sparse vectors on fixed support T with fixed sign pattern. ■

Theorem 8 establishes the existence of a positive bound $\rho_w^*(\alpha, p)$ and defines $\rho_w^*(\alpha, p)$ in (44). To obtain $\rho_w^*(\alpha, p)$, we first calculate $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ in Lemma 8 from (38) and $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ in Lemma 9 from (57) for every ρ , then find the largest $\rho_w^*(\alpha, p)$ such that (44) holds. We numerically calculate this bound and illustrate the results in Fig. 4 and Fig. 5. Fig. 4 shows the curve of $\rho_w^*(\alpha, p)$ against α for different p , and Fig. 5 shows the curve of $\rho_w^*(\alpha, p)$ against p for different α . When $\alpha \rightarrow 1$, $\rho_w^*(\alpha, p)$ goes to $2/3$ for all $p \in (0, 1)$, which coincides with the limiting threshold discussed in Section III-B. As indicated in Fig. 1.2 of [18], the weak recovery threshold of ℓ_1 -minimization is greater than $2/3$ for all α that is greater than 0.9 , since the weak recovery threshold of ℓ_p -minimization ($p \in [0, 1)$) when $\alpha \rightarrow 1$ is all $2/3$, therefore for all $\alpha > 0.9$, the weak recovery threshold of ℓ_1 -minimization is greater than that of ℓ_p -minimization for all $p \in [0, 1)$.

V. ℓ_1 -MINIMIZATION CAN PERFORM BETTER THAN ℓ_p -MINIMIZATION ($p \in [0, 1)$) FOR SPARSE

RECOVERY

For strong recovery, if ℓ_1 -minimization can recover all the k -sparse vectors, then ℓ_p -minimization is also guaranteed to recover all the k -sparse vectors for all $p \in [0, 1)$. However, this does not necessarily indicate that the performance of ℓ_p -minimization ($0 \leq p < 1$) is always better than that of ℓ_1 -minimization. Example 1 in Section II-B indicates that sometimes ℓ_1 -minimization can successfully recover the original

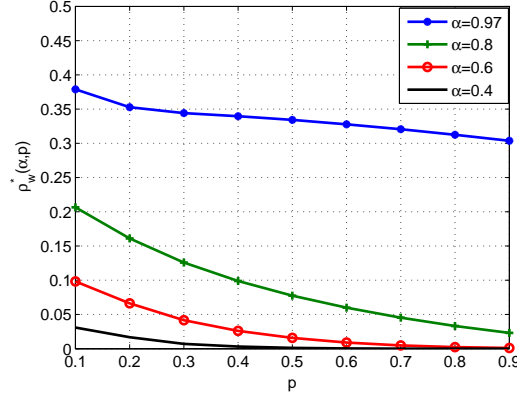


Fig. 5. $\rho_w^*(\alpha, p)$ against p for different α

sparse vector while ℓ_p -minimization ($p \in (0, 1)$) would return a vector that is denser than the original vector. Moreover, our results for weak recovery indicates that the performance of ℓ_1 -minimization is better than that of ℓ_p -minimization for all $p \in [0, 1)$ in at least the large α region ($\alpha > 0.9$).

We can roughly interpret the result as follows. Let $\alpha < 1$ be very close to 1, let n be large enough and A is a random Gaussian matrix. Then with overwhelming probability ℓ_1 -minimization can recover all the vectors up to $\rho_1 n$ -sparse and ℓ_p -minimization with some $p \in [0, 1)$ can recover all the vectors up to $\rho_2 n$ -sparse, and we know $\rho_1 < \rho_2$ from our discussion on strong bound. Note that since the limiting threshold of strong recovery via ℓ_p -minimization increases to 0.5 as p goes to 0, then we have $\rho_1 < \rho_2 \leq 0.5$. However, if we only consider the ability to recover all the vectors on one support with one sign pattern, with overwhelming probability ℓ_1 -minimization can recover vectors up to $\rho_3 n$ -sparse, while ℓ_p -minimization can recover vectors up to $\rho_4 n$ -sparse. From previous discussion about weak recovery threshold, we know that when α is very close to 1, $\rho_3 > \frac{2}{3} > \rho_4 > \frac{1}{2}$. Therefore we have $\rho_3 > \rho_4 > \rho_2 > \rho_1$. We illustrate the difference of ℓ_1 and ℓ_p -minimization in Fig. 6 and Fig. 7. Let Ω be the set of all $m \times n$ matrices with entries drawn from standard Gaussian distribution, and the probability measure $P(\Omega) = 1$. We pick $\rho \in (\rho_1, \rho_2)$ in Fig. 6. For a random measurement matrix A in Ω , since $\rho < \rho_3$, for any fixed support T with $|T| = \rho n$ and any fixed sign pattern σ_j , with high probability ℓ_1 -minimization can recover all the ρn -sparse vectors on T_i with sign pattern σ_j . Since we also have $\rho > \rho_1$, then with high probability strong recovery of ℓ_1 -minimization fails, in other words, ℓ_1 -minimization would fail to recover at least one vector with at most ρn non-zero entries. In Fig. 6 (a), $E_{T_i}^{\sigma_j}$ denotes the event that ℓ_1 -minimization can recover all the ρn -sparse vectors on support T_i with

sign patten σ_j . Then $P(E_{T_i}^{\sigma_j})$ is very close to 1 for every i and j . There are $\binom{n}{\rho n}$ different supports, and for each support, there are $2^{\rho n}$ different sign patterns. Let E denote the event that ℓ_1 -minimization can recover all the ρn -sparse vectors, then we have

$$E = \bigcap_{i \in \{1, \dots, \binom{n}{\rho n}\}, j \in \{1, \dots, 2^{\rho n}\}} E_{T_i}^{\sigma_j}.$$

Then although $P(E_{T_i}^{\sigma_j})$ is the same for all i and j and is very close to 1, $P(E)$ is close to 0, as indicated in Fig. 6 (a). For ℓ_p -minimization, since $\rho < \rho_2$, then with high probability, ℓ_p -minimization can recover all the ρn -sparse vectors. In Fig. 6 (b), \tilde{E} denotes the event that ℓ_p -minimization can recover all the ρn -sparse vectors, then

$$\tilde{E} = \bigcap_{i \in \{1, \dots, \binom{n}{\rho n}\}, j \in \{1, \dots, 2^{\rho n}\}} \tilde{E}_{T_i}^{\sigma_j},$$

where $\tilde{E}_{T_i}^{\sigma_j}$ denotes the event that ℓ_p -minimization recovers all the vectors on support T_i with sign pattern σ_j . In this case, $P(\tilde{E})$ is close to 1 as indicated in Fig. 6 (b). In Fig. 7, we pick $\rho \in (\rho_3, \rho_4)$. Then given any i and j , ℓ_1 -minimization can recover all the vectors on T_i with sign pattern σ_j with high probability, while ℓ_p -minimization fails to recover at least one vector on T_i with sign pattern σ_j with high probability. Therefore $P(E_{T_i}^{\sigma_j})$ is close to 1, while $P(\tilde{E}_{T_i}^{\sigma_j})$ is close to 0 for any given i and j . Therefore, if the sparse vectors we would like to recover are on one same support and share the same sign pattern, ℓ_1 -minimization can be a better choice than ℓ_p -minimization for all $p \in [0, 1)$ regardless of the amplitudes of the entries of a vector.

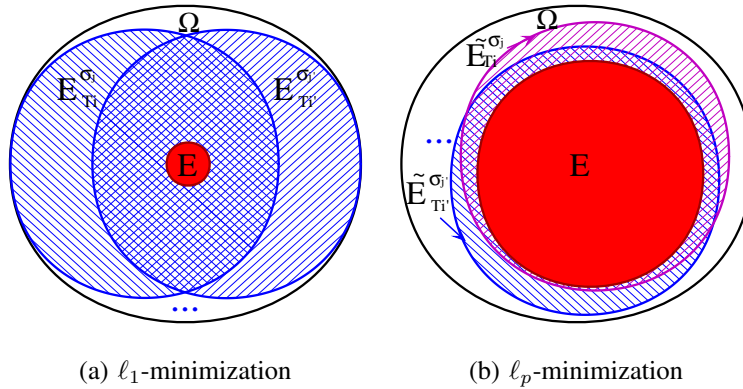


Fig. 6. Comparison of ℓ_1 and ℓ_p -minimization for $\rho \in (\rho_1, \rho_2)$.

To better understand how the recovery performance changes from strong recovery to weak recovery, let us consider another type of recovery: sectional recovery, which measures the ability of recovering all

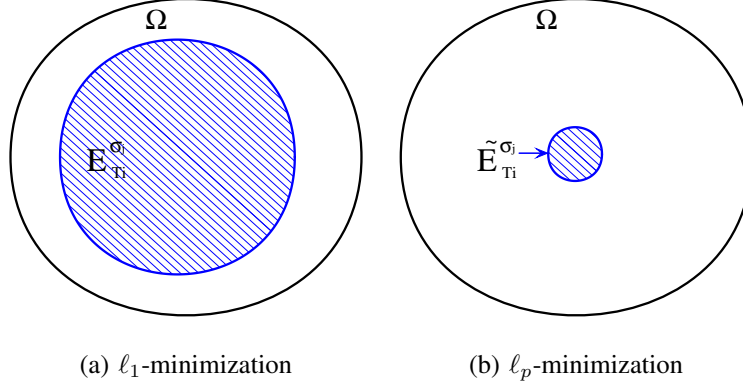


Fig. 7. Comparison of ℓ_1 and ℓ_p -minimization for $\rho \in (\rho_3, \rho_4)$.

the vectors on one support T . Therefore, the requirement for successful sectional recovery is stricter than that of weak recovery, but is looser than that of strong recovery. The necessary and sufficient condition of successful sectional recovery can be stated as:

Theorem 9. \mathbf{x} is the unique solution to ℓ_p -minimization problem ($p \in [0, 1]$) for all pn -sparse vector \mathbf{x} on some support T , if and only if

$$\|B_T \mathbf{z}\|_p^p < \|B_{T^c} \mathbf{z}\|_p^p \quad (46)$$

for all non-zero $\mathbf{z} \in \mathcal{R}^{n-m}$.

The difference of the null space condition for strong recovery and sectional recovery is that (46) should hold for every support T for strong recovery, but only needs to hold for one specific support T for sectional recovery. Though for strong recovery, if the null space condition holds for $p \in [0, 1]$, it also holds for all $q \in [0, p]$, this argument is not true for sectional recovery. Consider a simple example that the basis B of null space of A contains only one vector in \mathcal{R}^4 and $T = \{1, 2\}$. If $B = [16, 16, 1, 36]$, then one can check that $\|B_T\|_1 = 32 < 37 = \|B_{T^c}\|_1$, but $\|B_T\|_{0.5}^{0.5} = 8 > 7 = \|B_{T^c}\|_{0.5}^{0.5}$. If $B = [1, 4, 1, 9]$, then $\|B_T\|_1 < \|B_{T^c}\|_1$, and $\|B_T\|_{0.5}^{0.5} < \|B_{T^c}\|_{0.5}^{0.5}$. Therefore the null space condition of successful sectional recovery holds for p does not necessarily imply that it holds for another $q \neq p$.

Following the technique in Section III-B, one can show that when $\alpha \rightarrow 1$ and n is large enough, the recovery threshold of sectional recovery is $1/2$ for all $p \in [0, 1]$. We skip the proof here as it follows the lines in Section III-B. To summarize, regarding the recovery threshold when $\alpha \rightarrow 1$, ℓ_p -minimization ($p \in [0, 1]$) has a higher threshold for smaller p for strong recovery; the threshold is all $1/2$ for all $p \in [0, 1]$ for sectional recovery; and the threshold is all $2/3$ for $p \in [0, 1]$ and 1 for $p = 1$ for weak

recovery. We can see how recovery performance changes when the requirement for successful recovery changes from strong to weak.

VI. NUMERICAL EXPERIMENTS

We present the results of numerical experiments to explore the performance of ℓ_p -minimization. As mentioned earlier, (3) is indeed non-convex and it is hard to compute its global minimum. Here we employ the iteratively reweighted least squares algorithm [11][12] to compute the local minimum of (3), please refer to [12] about the details of the algorithm.

Example 2. ℓ_p -minimization using IRLS [12]

We fix $n = 200$ and $m = 100$, and increase ρ from 0.01 to 0.5 as a percentage of n . For each ρ , we repeat the following procedure 100 times. We first generate a n -dimensional vector \mathbf{x} with ρn nonzero entries. The location of the non-zero entries are chosen randomly, and each non-zero value follows from standard Gaussian distribution. We then generate a $m \times n$ matrix A with i.i.d. $\mathcal{N}(0, 1)$ entries. We let $\mathbf{y} = A\mathbf{x}$ and run the iteratively reweighted least squares algorithm to search for a local minimum of (3) with p chosen to be 0.2, 0.5, and 0.8 respectively. Let \mathbf{x}^* be the output of the algorithm, if $\|\mathbf{x}^* - \mathbf{x}\|_2 \leq 10^{-4}$, we say the recovery of \mathbf{x} is the successful. Figure 8 records the percentage of times that the recovery is successful for different sparsity ρn . Note that the iteratively reweighted least squares algorithm is designed to obtain a local minimum of the ℓ_p -minimization problem (3), and is not guaranteed to obtain the global minimum. However, as shown in Figure 8, it indeed recovers the sparse vectors up to certain sparsity. For $\ell_{0.2}$, $\ell_{0.5}$ and $\ell_{0.8}$ -minimization computed by the heuristic, the sparsity ratios of successful recovery are 0.025, 0.024, and 0.015 respectively.

Example 3. Strong recovery vs. weak recovery

We also compare the performance of ℓ_p -minimization and ℓ_1 -minimization both for strong recovery in Fig. 9 and for weak recovery in Fig. 10 when α is large. We employ CVX [24] to solve ℓ_1 -minimization and still employ the iteratively reweighted least squares algorithm to compute a local minimum of ℓ_p -minimization. We fix $n = 50$ and $m = 48$ and independently generate one hundred random matrices $A^{m \times n}$ with i.i.d. $\mathcal{N}(0, 1)$ entries and evaluate the performance of strong recovery and weak recovery. For each matrix, we increase ρ from 0.04 to 1. In weak recovery, we consider recovering nonnegative vectors on support $T = \{1, \dots, \rho n\}$. For a given ρ , we generate one hundred and fifty vectors and claim the weak recovery of ρn -sparse vectors to be successful if and only if all the vectors are successfully recovered. For each vector \mathbf{x} , x_i ($i \in T$) is generated from $\mathcal{N}(0, 1)$ with probability 0.5, and $\mathcal{N}(1000, 1)$ with probability 0.5. As discussed in Section II, the condition for successful weak recovery via ℓ_1 -

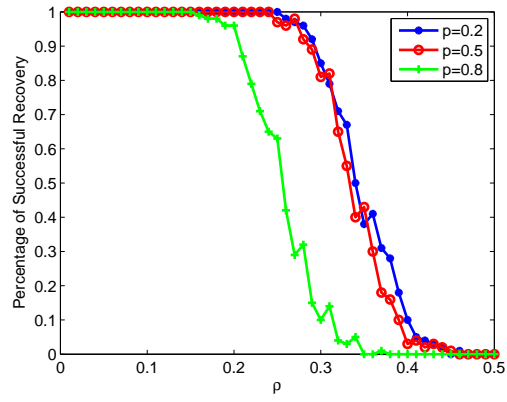


Fig. 8. Successful recovery of ρn -sparse vectors via ℓ_p -minimization

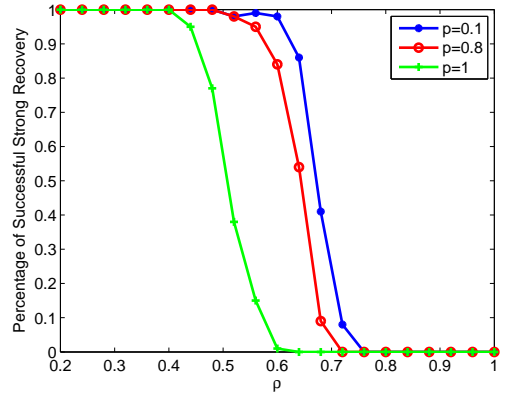


Fig. 9. Successful strong recovery of ρn -sparse vectors

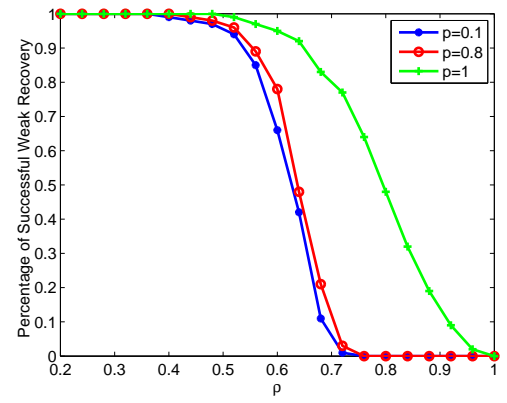


Fig. 10. Successful weak recovery of ρn -sparse vectors

minimization is the same for every nonnegative vector on T , therefore if ℓ_1 -minimization recovers all the vectors we generated, it should also recover all the nonnegative vectors on T . ℓ_p -minimization ($p \in [0, 1)$), on the other hand, can recover some nonnegative vectors on T while at the same time fails to recover some other nonnegative vectors on T . Therefore, since we could not check every nonnegative \mathbf{x} on T , ℓ_p -minimization ($p < 1$) can still fail to recover some other nonnegative vector on T even if we declare the weak recovery to be “successful”. In strong recovery, for each ρ , we generate two hundred vectors and claim the strong recovery to be successful if and only if all these vectors are correctly recovered. To generate a ρn -sparse vector \mathbf{x} , we first randomly pick a support T with $|T| = \rho n$. For each x_i ($i \in T$), x_i is generated from $\mathcal{N}(0, 1)$ with probability 0.5, from $\mathcal{N}(1000, 1)$ with probability 0.25, and from $\mathcal{N}(-1000, 1)$ with probability 0.25. The average performance of one hundred random matrices for strong recovery is plotted in Fig. 9, and the average performance of weak recovery is plotted in Fig. 10. Note that we only apply iteratively reweighted least squares algorithm to approximate the performance of ℓ_p -minimization, therefore the solution returned by the algorithm may not always be the solution of ℓ_p -minimization. Simulation results indicate that for strong recovery, the recovery threshold increases as p decreases, while for the weak recovery, interestingly, the recovery threshold of ℓ_1 -minimization is higher than any other ℓ_p -minimization for $p < 1$.

VII. CONCLUSION

This paper analyzes the ability of ℓ_p -minimization ($0 \leq p \leq 1$) to recover high-dimensional sparse vectors from low-dimensional linear measurements where the measurement matrix $A^{m \times n}$ has i.i.d. standard Gaussian entries. When $\alpha = m/n \rightarrow 1$, we provide a tight threshold $\rho^*(p)$ of the sparsity ratio separating the success and failure of strong recovery which requires to recover all the sparse vectors. $\rho^*(p)$ strictly decreases from 0.5 to 0.239 as p increases from 0 to 1. For weak recovery which only needs to recover sparse vectors on some support with some sign pattern, we first provide an equivalent null space characterization of successful weak recovery, then prove that the threshold of sparsity ratio separating the success and failure of ℓ_p -minimization is $2/3$ for all $p < 1$, compared with the threshold 1 for ℓ_1 -minimization. For any $\alpha < 1$, we provide a bound $\rho^*(\alpha, p)$ of sparsity ratio below which strong recovery via ℓ_p -minimization succeeds with overwhelming probability, and our bound $\rho^*(\alpha, p)$ improves on the existing bounds in the large α region. We also provide a bound $\rho_w^*(\alpha, p)$ of sparsity ratio below which weak recovery succeeds with overwhelming probability.

Throughout the paper, we assume that the measurements $\mathbf{y} = A\mathbf{x}$ are exact, and it would be interesting to consider the case that the measurements are noisy, i.e. $\mathbf{y} = A\mathbf{x} + \mathbf{e}$ where \mathbf{e} is the vector of noise.

Moreover, we assume that \mathbf{x} is exactly sparse, i.e. most of its entries are exactly zero. The extension of results to approximately sparse vectors whose coefficients (if ordered) decay rapidly is also worth pursuit.

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REFERENCES

- [1] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A simple proof of the restricted isometry property for random matrices," *Constructive Approximation*, vol. 28, pp. 253–263, 2008.
- [2] R. Berinde, A. Gilbert, P. Indyk, H. Karloff, and M. Strauss, "Combining geometry and combinatorics: a unified approach to sparse signal recovery," *Preprint*, 2008.
- [3] R. Berinde and P. Indyk, "Sparse recovery using sparse random matrices," *MIT-CSAIL Technical Report*, 2008.
- [4] J. Blanchard, C. Cartis, and J. Tanner, "The restricted isometry property and ℓ^q regularization: phase transitions for sparse approximation," *Preprint*, 2009.
- [5] A. Bruckstein, M. Elad, and M. Zibulevsky, "On the uniqueness of nonnegative sparse solutions to underdetermined systems of equations," *IEEE Trans. Inf. Theory*, vol. 54, no. 11, pp. 4813–4820, Nov. 2008.
- [6] E. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 489 – 509, Feb. 2006.
- [7] E. Candès and T. Tao, "Decoding by linear programming," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4203–4215, Dec. 2005.
- [8] —, "Near-optimal signal recovery from random projections: Universal encoding strategies?" *IEEE Trans. Inf. Theory*, vol. 52, no. 12, pp. 5406–5425, Dec. 2006.
- [9] R. Chartrand, "Exact reconstruction of sparse signals via nonconvex minimization," *Signal Process. Lett.*, vol. 14, no. 10, pp. 707–710, 2007.
- [10] —, "Nonconvex compressed sensing and error correction," in *Proc. ICASSP*, 2007.
- [11] R. Chartrand and W. Yin, http://www.caam.rice.edu/~wy1/paperfiles/TR08-01/IRLS_CS.rar.
- [12] —, "Iteratively reweighted algorithms for compressive sensing," in *Proc. IEEE ICASSP 2008.*, Apr. 2008, pp. 3869–3872.
- [13] A. Cohen, W. Dahmen, and R. DeVore, "Compressed sensing and best k-term approximation," *Journal of the American Mathematical Society*, vol. 22, pp. 211–231, 2009.
- [14] M. E. Davies and R. Gribonval, "Restricted isometry constants where l_p sparse recovery can fail for $0 < p \leq 1$," *IEEE Trans. Inf. Theory*, vol. 55, no. 5, pp. 2203–2214, 2009.
- [15] D. Donoho, "High-dimensional centrally symmetric polytopes with neighborliness proportional to dimension," *Discrete Comput. Geom.*, 2006.
- [16] D. L. Donoho and J. Tanner, "Sparse nonnegative solution of underdetermined linear equations by linear programming," in *Proc. Natl. Acad. Sci. U.S.A.*, vol. 102, no. 27, 2005, pp. 9446–9451.
- [17] —, "Counting the faces of randomly-projected hypercubes and orthants, with applications," *Journal of the American Mathematical Society*, vol. 22, no. 1, pp. 1–53, 2009.
- [18] D. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, April 2006.

- [19] D. Donoho and X. Huo, “Uncertainty principles and ideal atomic decomposition,” *IEEE Trans. Inf. Theory*, vol. 47, no. 7, pp. 2845 – 2862, Nov. 2001.
- [20] C. Dwork, F. McSherry, and K. Talwar, “The price of privacy and the limits of lp decoding,” in *Proc. STOC*, 2007, pp. 85–94.
- [21] M. Elad and A. Bruckstein, “A generalized uncertainty principle and sparse representation in pairs of bases,” *IEEE Trans. Inf. Theory*, vol. 48, no. 9, pp. 2558 – 2567, Sep. 2002.
- [22] S. Foucart and M.-J. Lai, “Sparsest solutions of underdetermined linear systems via l_q -minimization for $0 < q \leq 1$,” *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 395 – 407, 2009.
- [23] J.-J. Fuchs, “On sparse representations in arbitrary redundant bases,” *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 1341 – 1344, Jun. 2004.
- [24] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 1.21,” <http://cvxr.com/cvx>, Oct. 2010.
- [25] R. Gribonval and M. Nielsen, “Sparse representations in unions of bases,” *IEEE Trans. Inf. Theory*, vol. 49, no. 12, pp. 3320 – 3325, Dec. 2003.
- [26] —, “Highly sparse representations from dictionaries are unique and independent of the sparseness measure,” *Applied and Computational Harmonic Analysis*, vol. 22, no. 3, pp. 335 – 355, 2007.
- [27] J. Haupt and R. Nowak, “Signal reconstruction from noisy random projections,” *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 4036 – 4048, Sep. 2006.
- [28] M. Ledoux, Ed., *The Concentration of Measure Phenomenon*. American Mathematical Society, 2001.
- [29] R. Saab, R. Chartrand, and O. Yilmaz, “Stable sparse approximations via nonconvex optimization,” in *Proc. ICASSP*, 2008.
- [30] M. Stojnic, “A simple performance analysis of ℓ_1 optimization in compressed sensing,” in *Proc. IEEE ICASSP 2009.*, Apr. 2009, pp. 3021 – 3024.
- [31] M. Stojnic, W. Xu, and B. Hassibi, “Compressed sensing - probabilistic analysis of a null-space characterization,” in *Proc. ICASSP*, 2008, pp. 3377–3380.
- [32] J. Wright and Y. Ma, “Dense error correction via l_1 -minimization,” in *Proc. ICASSP 2009.*, Apr. 2009, pp. 3033 – 3036.
- [33] W. Xu and B. Hassibi, “Efficient compressive sensing with deterministic guarantees using expander graphs,” in *Information Theory Workshop, 2007. ITW '07. IEEE*, Sept. 2007, pp. 414–419.
- [34] —, “Compressed sensing over the Grassmann manifold: A unified analytical framework,” in *Proc. Allerton 2008*, Sep. 2008, pp. 562 – 567.
- [35] —, “Compressive sensing over the Grassmann manifold: a unified geometric framework,” *Preprint*, 2010.
- [36] Y. Zhang, “When is missing data recoverable,” Tech. Rep., 2006.

APPENDIX

A. Calculation of $\lambda_{\max}(\alpha, p)$ in Lemma 7

Define $c_{\max} = \frac{1}{n} \max_{\mathbf{z} \in \mathcal{S}} \|B\mathbf{z}\|_p^p$, then for any non-zero vector \mathbf{z} , $\|B\mathbf{z}\|_p^p \leq \|\mathbf{z}\|_p^p c_{\max} n$. Let Σ_1 be a γ -net of \mathcal{S} with cardinality at most $(1 + 2/\gamma)^{n-m}$ [28] and $\gamma > 0$ to be chosen later, and define

$$\eta = \frac{1}{n} \max_{\mathbf{z} \in \Sigma_1} \|B\mathbf{z}\|_p^p.$$

Then from the definition of γ -net, for every $\mathbf{z} \in \mathcal{S}$, there exists $\mathbf{z}' \in \Sigma_1$ such that $\|\mathbf{z} - \mathbf{z}'\|_2 \leq \gamma$. Note that for every $\mathbf{z} \in \mathcal{S}$, $\|B\mathbf{z}\|_p^p \leq \|B\mathbf{z}'\|_p^p + \|B(\mathbf{z} - \mathbf{z}')\|_p^p \leq \eta n + \gamma^p c_{\max} n$. Then $c_{\max} n \leq \eta n + \gamma^p c_{\max} n$, which leads to

$$c_{\max} \leq \eta / (1 - \gamma^p). \quad (47)$$

To characterize c_{\max} , we first characterize η . We will show that there exists a constant $a > E[|X|^p]$ where $X \sim \mathcal{N}(0, 1)$ such that with overwhelming probability, $\|B\mathbf{z}\|_p^p < an$ for all \mathbf{z} in Σ_1 . Given $\mathbf{z} \in \mathcal{S}$, $B_i \mathbf{z}$ ($i = 1, \dots, n$) are i.i.d. $\mathcal{N}(0, 1)$ random variables where B_i is the i^{th} row of B . Then

$$\begin{aligned} P(\eta \geq a) &= P(\exists \mathbf{z} \in \Sigma_1 \text{ s.t. } \|B\mathbf{z}\|_p^p \geq an) \\ &\leq \sum_{\mathbf{z} \in \Sigma_1} P(\|B\mathbf{z}\|_p^p \geq an) \\ &\leq (1 + 2/\gamma)^{n-m} \min_{t>0} e^{-tan} E[e^{t \sum_i |B_i \mathbf{z}|^p}] \\ &= (1 + 2/\gamma)^{(1-\alpha)n} \min_{t>0} e^{-tan} E[e^{t|X|^p}]^n \\ &= e^{((1-\alpha) \log(1+2/\gamma) + \min_{t>0} (\log(E[e^{t|X|^p}]) - at))n}, \end{aligned} \quad (48)$$

where $X \sim \mathcal{N}(0, 1)$, the first inequality follows from the union bound, and the second inequality follows from the Chernoff bound.

Since the second-order derivative of $\log(E[e^{t|X|^p}]) - at$ to t is positive, then its minimum is achieved where its first-order derivative is 0. To calculate the value of t where the minimum is achieved, we have

$$\begin{aligned} 0 &= \frac{d[\log(E[e^{t|X|^p}]) - at]}{dt} \\ &= \frac{d}{dt} (\log(\sqrt{\frac{2}{\pi}} \int_0^\infty e^{tx^p - \frac{1}{2}x^2} dx) - at) \\ &= \frac{\int_0^\infty x^p e^{tx^p - \frac{1}{2}x^2} dx}{\int_0^\infty e^{tx^p - \frac{1}{2}x^2} dx} - a. \end{aligned} \quad (49)$$

Note that when $a > E[|X|^p]$, the solution of t to (49) is always positive, thus it is also the solution to $\min_{t>0} (\log(E[e^{t|X|^p}]) - at)$. One can check that for any γ , the exponent in (48) is negative when a is large enough. To see this, let $t = 2(1-\alpha) \log(1+2/\gamma)/a$, then $\log(E[e^{t|X|^p}]) - at$ goes to $-2(1-\alpha) \log(1+2/\gamma)$ as a goes to infinity. Thus, when a is sufficiently large, $\log(E[e^{t|X|^p}]) - at < -(1-\alpha) \log(1+2/\gamma)$ if $t = c/a$. Therefore, the exponent in (48) is negative when a is large enough. Thus, we can pick $a(\alpha, p, \gamma)$ large enough such that there exists some constant $c_{12} > 0$ and $P(\eta \geq a(\alpha, p, \gamma)) \leq e^{-c_{12}n}$ holds. Then

$$P(c_{\max} \geq \frac{a(\alpha, p, \gamma)}{1 - \gamma^p}) \leq P(\frac{\eta}{1 - \gamma^p} \geq \frac{a(\alpha, p, \gamma)}{1 - \gamma^p}) \leq e^{-c_{12}n},$$

where the first inequality follows from (47). Let

$$\lambda_{\max}(\alpha, p) = \min_{\gamma} a(\alpha, p, \gamma)/(1 - \gamma^p), \quad (50)$$

then there exists $c_{12}(\alpha, p, \lambda_{\max}) > 0$ such that with probability at least $1 - e^{-c_{12}n}$, for every $\mathbf{z} \in \mathcal{S}$, $\|B\mathbf{z}\|_p^p < \lambda_{\max}n$. Thus, Lemma 7 follows.

B. Proof of Lemma 8

Proof: Define $c'_{\max} = \frac{1}{(1-\rho)n} \max_{\mathbf{z} \in \mathcal{S}} \|B_{T^c} \mathbf{z}\|_p^p$. Let Σ_4 be a γ -net of \mathcal{S} with cardinality at most $(1 + 2/\gamma)^{n-m}$ and γ being the value where $\lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)$ is achieved, and define

$$\eta' = \frac{1}{(1-\rho)n} \max_{\mathbf{z} \in \Sigma_4} \|B\mathbf{z}\|_p^p.$$

Then same as that in the calculation of $\lambda_{\max}(\alpha, p)$ in Appendix-A, we have

$$c'_{\max} \leq \eta'/(1 - \gamma^p).$$

We use λ_{\max} to denote $\lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)$ for simplicity. We first show that with overwhelming probability, $\|B_{T^c} \mathbf{z}\|_p^p < (1 - \rho)\lambda_{\max}n$ for all \mathbf{z} in \mathcal{S} , or equivalently $c'_{\max} < \lambda_{\max}$. Note that

$$\begin{aligned} & P(c'_{\max} \geq \lambda_{\max}) \\ & \leq P(\eta'/(1 - \gamma^p) \geq \lambda_{\max}) \\ & = P(\exists \mathbf{z} \in \Sigma_4 \text{ s.t. } \|B_{T^c} \mathbf{z}\|_p^p \geq (1 - \rho)\lambda_{\max}(1 - \gamma^p)n) \\ & \leq \sum_{\mathbf{z} \in \Sigma_4} P(\|B_{T^c} \mathbf{z}\|_p^p \geq (1 - \rho)\lambda_{\max}(1 - \gamma^p)n) \\ & \leq (1 + \frac{2}{\gamma})^{n-m} \min_{t>0} \frac{E[e^{t \sum_{i \in T^c} |B_i \mathbf{z}|^p}]}{e^{t(1-\rho)\lambda_{\max}(1-\gamma^p)n}} \\ & = (1 + \frac{2}{\gamma})^{(1-\alpha)n} \min_{t>0} \frac{E[e^{t|X|^p}]^{(1-\rho)n}}{e^{t(1-\rho)\lambda_{\max}(1-\gamma^p)n}} \\ & = e^{(1-\rho)n(\frac{1-\alpha}{1-\rho} \log(1+\frac{2}{\gamma}) + \min_{t>0} (\log(E[e^{t|X|^p}]) - \lambda_{\max}(1-\gamma^p)t))}, \end{aligned} \quad (51)$$

where $X \sim \mathcal{N}(0, 1)$. From the definition of $\lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)$, and that γ is chosen to be the value where $\lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)$ is achieved, we know that there exists $c_{13} > 0$ such that (51) $\leq e^{-c_{13}n}$. Therefore it holds with probability at least $1 - e^{-c_{13}n}$ that for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c} \mathbf{z}\|_p^p < (1 - \rho)\lambda_{\max}n$.

Similarly, define $c'_{\min} = \frac{1}{(1-\rho)n} \min_{\mathbf{z} \in \mathcal{S}} \|B\mathbf{z}\|_p^p$. Let Σ_5 be a γ -net of \mathcal{S} with cardinality at most $(1 + 2/\gamma)^{n-m}$ and γ being the value where $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ is achieved, note that

$$\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p) = \gamma^{p(\frac{1-\alpha}{1-\rho} + \epsilon)} - \gamma^p \lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)$$

for some $\epsilon \in (0, \frac{1-\alpha}{1-\rho})$ according to the definition of $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$. We use λ_{\min} and λ_{\max} to denote $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ and $\lambda_{\max}(\frac{\alpha-\rho}{1-\rho}, p)$ for simplicity. We define

$$\theta' = \frac{1}{(1-\rho)n} \min_{\mathbf{z} \in \Sigma_5} \|B_{T^c} \mathbf{z}\|_p^p.$$

Like in the calculation of $\lambda_{\min}(\alpha, p)$ in Section IV-A1, we have

$$c'_{\min} \geq \theta' - \gamma^p c'_{\max}.$$

We next show that with overwhelming probability, $\|B_{T^c} \mathbf{z}\|_p^p > (1-\rho)\lambda_{\min}n$ for all \mathbf{z} in \mathcal{S} , or equivalently $c'_{\min} > \lambda_{\min}$. Note that

$$\begin{aligned} & P(c'_{\min} \leq \lambda_{\min}) \\ &= P(c'_{\min} \leq \gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)} - \gamma^p \lambda_{\max}) \\ &\leq P(\theta' - \gamma^p c'_{\max} \leq \gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)} - \gamma^p \lambda_{\max}) \\ &\leq P(\theta' \leq \gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)}) + P(c'_{\max} \geq \lambda_{\max}) \\ &\leq P(\theta' \leq \gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)}) + e^{-c_{13}n}, \end{aligned} \tag{52}$$

where the last inequality follows from (51). To calculate $P(\theta' \leq \gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)})$, note that

$$\begin{aligned} & P(\theta' \leq \gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)}) \\ &= P(\exists \mathbf{z} \in \Sigma_5 \text{ s.t. } \|B_{T^c} \mathbf{z}\|_p^p \leq (1-\rho)\gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)}n) \\ &\leq \sum_{\mathbf{z} \in \Sigma_5} P(\sum_{i \in T^c} |B_i \mathbf{z}|^p \leq (1-\rho)\gamma^{p(\frac{1-\alpha}{1-\rho}+\epsilon)}n) \\ &\leq (1 + \frac{2}{\gamma})^{(1-\alpha)n} e^{(1-\rho)n} E[e^{-\gamma^{-p(\frac{1-\alpha}{1-\rho}+\epsilon)} |X|^p}]^{(1-\rho)n} \\ &= e^{(1-\rho)n(\frac{1-\alpha}{1-\rho} \log(1+\frac{2}{\gamma}) + \log(E[e^{-\gamma^{-p(\frac{1-\alpha}{1-\rho}+\epsilon)} |X|^p]})) + 1} \\ &= e^{(1-\rho)n(\frac{1-\alpha}{1-\rho} \log(1+\frac{2}{\gamma}) + \log(O(\gamma^{\frac{1-\alpha}{1-\rho}+\epsilon}))) + 1}, \end{aligned} \tag{53}$$

where $X \sim \mathcal{N}(0, 1)$, the second inequality follows from the Chernoff bound, and the last equality follows from (35). Since γ is chosen to be the value where $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$ is achieved, then according to the definition of $\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)$, (53) $\leq e^{-\kappa n}$ for some positive $\kappa > 0$. Thus, from (52) we have

$$P(c'_{\min} \leq \lambda_{\min}) \leq e^{-\kappa n} + e^{-c_{13}n} \leq e^{-c_{14}n},$$

for some $c_{14} > 0$. Then, with probability at least $1 - e^{-c_{14}n}$, for all $\mathbf{z} \in \mathcal{S}$, $\|B_{T^c} \mathbf{z}\|_p^p > (1-\rho)\lambda_{\min}(\frac{\alpha-\rho}{1-\rho}, p)n$. ■

C. Calculation of $\tilde{\lambda}_{\max}(\alpha, p, \rho)$ in Lemma 9

Proof: Define $\tilde{c}_{\max} = \frac{1}{\rho n} \max_{\mathbf{z} \in \mathcal{S}} \|B_T - \mathbf{z}\|_p^p$. Let Σ_6 be a γ -net of \mathcal{S} with cardinality at most $(1 + 2/\gamma)^{n-m}$ and $\gamma > 0$ to be chosen later, and define $\tilde{\eta} = \frac{1}{\rho n} \max_{\mathbf{z} \in \Sigma_4} \|B_T - \mathbf{z}\|_p^p$. Then from (25), for any $\mathbf{z} \in \mathcal{S}$, $\mathbf{z} = \sum_{j \geq 0} \gamma_j \mathbf{v}_j$ hold, where $\gamma_0 = 1$, $\gamma_j \leq \gamma^j$ and $\mathbf{v}_j \in \Sigma_6$. From (26) we have

$$\begin{aligned} \|B_T - \mathbf{z}\|_p^p &\leq \sum_{j \geq 0} \gamma^{jp} \sum_{i \in T: (B_i \mathbf{v}_j)_{x_i} < 0} |B_i \mathbf{v}_j|^p \\ &\leq \sum_{j \geq 0} \gamma^{jp} \tilde{\eta} \rho n \\ &\leq \tilde{\eta} \rho n / (1 - \gamma^p) \end{aligned} \quad (54)$$

Since (54) holds for every $\mathbf{z} \in \mathcal{S}$, then $\tilde{c}_{\max} \rho n \leq \tilde{\eta} \rho n / (1 - \gamma^p)$, which leads to $\tilde{c}_{\max} \leq \tilde{\eta} / (1 - \gamma^p)$. Define a random variable S_i for each i in T that is equal to 1 if $B_i \mathbf{z} x_i < 0$ and equal to 0 otherwise. Then $\|B_T - \mathbf{z}\|_p^p = \sum_{i \in T} |B_i \mathbf{z}|^p S_i$. Then for any \tilde{a} ,

$$\begin{aligned} P(\tilde{c}_{\max} \geq \frac{\tilde{a}}{1 - \gamma^p}) &\leq P(\frac{\tilde{\eta}}{1 - \gamma^p} \geq \frac{\tilde{a}}{1 - \gamma^p}) \\ &= P(\tilde{\eta} \geq \tilde{a}) = P(\exists \mathbf{z} \in \Sigma_6 \text{ s.t. } \|B_T - \mathbf{z}\|_p^p \geq \tilde{a} \rho n) \\ &\leq \sum_{\mathbf{z} \in \Sigma_6} P(\|B_T - \mathbf{z}\|_p^p \geq \tilde{a} \rho n) \\ &= (1 + \frac{2}{\gamma})^{n-m} P(\sum_{i \in T} |B_i \mathbf{z}|^p S_i \geq \tilde{a} \rho n) \\ &\leq (1 + \frac{2}{\gamma})^{(1-\alpha)n} \min_{t > 0} \frac{E[e^{t|X|^p S}]^{\rho n}}{e^{t \tilde{a} \rho n}} \\ &= e^{((1-\alpha) \log(1 + \frac{2}{\gamma}) + \rho \min_{t > 0} (\log(E[e^{t|X|^p S}]) - \tilde{a} t))n}, \end{aligned} \quad (55)$$

where $X \sim \mathcal{N}(0, 1)$, $S = 1$ if $X < 0$ and $S = 0$ otherwise.

Since the second derivative of $\log(E[e^{t|X|^p S}]) - \tilde{a} t$ to t is positive, then its minimum is achieved where its first derivative is 0. To calculate the value of t where the minimum is achieved, we have

$$\begin{aligned} 0 &= \frac{d[\log(E[e^{t|X|^p S}]) - \tilde{a} t]}{dt} \\ &= \frac{d}{dt} (\log(\sqrt{\frac{1}{2\pi}} \int_0^\infty e^{tx^p - \frac{1}{2}x^2} dx + \frac{1}{2}) - \tilde{a} t) \\ &= \frac{\int_0^\infty x^p e^{tx^p - \frac{1}{2}x^2} dx}{\int_0^\infty e^{tx^p - \frac{1}{2}x^2} dx + \sqrt{\pi/2}} - \tilde{a}. \end{aligned} \quad (56)$$

Note that when $\tilde{a} > E[|X|^p S]$, the solution of t to (56) is always positive, thus it is also the solution to $\min_{t > 0} (\log(E[e^{t|X|^p S}]) - \tilde{a} t)$. Given any ρ and γ , when \tilde{a} is large enough, the exponent in (55) is

negative. We can pick $\tilde{a}(\alpha, p, \rho, \gamma)$ as small as possible while still keeping the exponent in (55) negative.

Let

$$\tilde{\lambda}_{\max}(\alpha, p, \rho) = \min_{\gamma} \frac{\tilde{a}(\alpha, p, \rho, \gamma)}{1 - \gamma^p}, \quad (57)$$

then there exists $c_{15} > 0$ such that with probability at least $1 - e^{-c_{15}n}$, $c_{\max} < \tilde{\lambda}_{\max}(\alpha, p, \rho)$, or equivalently, for every $\mathbf{z} \in \mathcal{S}$, $\|B_T - \mathbf{z}\|_p^p < (1 - \rho)\tilde{\lambda}_{\max}(\alpha, p, \rho)n$. Thus, Lemma 9 follows. ■